Active noise control
Chapter I - Linear acoustics

EDEE 2011 - Dr. Hervé Lissek
Laboratoire d’Electromagnétisme et d’Acoustique
Outline of the chapter

- Propagation of sound
- Sound sources - radiation
- Interferences of sound
- Refraction of sound
- Guided waves in 1D

- *Practical work (COMSOL MULTIPHYSICS)*

- Radiation of complex structures (pistons)
- Noise control
Linear acoustics
Propagation of sound
Hypothesis

Linear acoustics:
- compressible fluid (liquid or gas)
- homogeneous
- continuous
- isotropic
- isotherm and isobar without the acoustic disturbance
- no external force (like weight), no flow
- unlimited medium,
- small movements hypothesis (allows the linearization of equations)
- no energy loss, by thermal dissipation
Definitions

• **Acoustic Field**: medium in which the *acoustic wave* propagates

• Characterized by *physical quantities* function of *time* and *space*: *acoustical quantities*

• **Particle**: portion of fluid which dimensions are much bigger than molecules, but small in order to assume small variations of acoustical quantities
Definitions

- **Acoustic displacement:** \( \xi(t, r_s) = r(t) - r_s \)  \( \text{m} \)

- **Acoustic velocity:** \( v(t, r) = \frac{\partial \xi}{\partial t} \)  \( \text{m.s}^{-1} \)

- **Acoustic pressure:** \( p(t, r) = p'(t) - p_s \)  \( \text{Pa} \)

- **Condensation:** \( s(t, r) = \frac{[\rho'(t, r) - \rho]}{\rho} \)  \( \text{l} \)
Physical laws

1) Newton’s law: governs the particles’ movement
2) Mass conservation (constant mass)
3) Compressibility law: governs their deformation

By linearizing and combining these laws, we obtain the sound propagation equation
Physical laws

General case: propagation in the 3D space

**I - Newton**

\[ F = m \ddot{v} \]

where \( F_{x} = \left[ \frac{p(x) - p(x + dx)}{-\partial p / \partial x dx} \right] \cdot dS = -\partial_{x} p . dV \)

\[ F = -V . \nabla p \]
Physical laws

The hypothesis of small movements leads to neglect the $2^{\text{nd}}$ order terms.

If we don’t take account of differences of sections of parallel faces, neither of deformations of the particle: mass does not vary much.

$$\rho'(t, r) = \frac{m}{V(t, r)} \approx \rho$$

Newton: $\nabla p = -\rho \ddot{v} = -\rho \partial_t v$ (local acceleration)

if $\partial_x v_x \cdot v_x + \partial_y v_y \cdot v_y + \partial_z v_z \cdot v_z = (v \cdot \nabla)v = 0$

(convective acceleration neglected)
Physical laws

2-Continuity equation

Constant mass

\[ V'(t, r) \cdot \rho'(t, r) = m \]

\[ \left( \frac{\delta V}{V'} \right) + \left( \frac{\delta \rho}{\rho'} \right) = 0 \]

Definition of condensation \( s \):

\[ s = - \left( \frac{\delta V}{V} \right) \]

where \( V \) is the particle volume at rest \((dx. dy. dz)\)

\[ \delta V_x = \delta \xi_x \cdot dy. dz = \partial_x \xi_x \cdot V \]

then \( \delta V = V \nabla \cdot \xi \) continuity equation

and \( \partial_t s + \nabla \cdot \nu = 0 \)
Physical laws

3-Local form of compressibility

let’s consider a volume $V$ of compressible fluid, subject to the pressure variation $\delta p$

$\Rightarrow$ its volumes varies of $\delta V$

Compressibility law, for a loss-less medium:

$\delta p = -K \frac{\delta V}{V}$  \hspace{1cm} $K$: compressibility

Hypothesis of small movements: $K$ independant of $\delta V$ and $\delta p$

(but depending of the static state of the fluid)

Local form of compressibility: $p = K \cdot s$
Wave equation

By combining the 3 physical laws with an emphasis on the acoustic pressure:

$$\nabla^2 p - \frac{\rho}{K} \partial_t^2 p = 0$$

$$c = (\rho / K)^{1/2}$$

d’Alembert’s equation

$$\nabla^2 p - \frac{1}{c^2} \partial_t^2 p = 0$$
Wave equation

On the acoustic velocity

\[ \nabla^2 v + \nabla \times (\nabla \times v) - \frac{1}{c^2} \partial^2_t v = 0 \]

In case of a longitudinal wave

\[ \nabla^2 v - \frac{1}{c^2} \partial^2_t v = 0 \]
Velocity potential

In order to analytically solve the propagation problems, let’s introduce a velocity potential concept $\Phi \,[m^2.s^{-1}]$ such as:

$$\nu = -\nabla \Phi$$

(the rotational of a gradient being null, rotational part of the velocity is therefore null)

- It is a scalar value
- It has no physical signification
- All the acoustic quantities derive from $\Phi$. 
Example: 1D sound generation/propagation
Example: 1D sound generation/propagation
Solutions of the wave equation

- Let’s consider the acoustic field quantity $\Phi$
- In the framework of linear acoustics, every field quantity is described by harmonic fluctuations (i.e. $e^{i\omega t}$)
- Then
  - by deriving twice over time:
    \[
    \partial_t^2 \Phi = -\omega^2 \Phi
    \]
  - d‘Alambert equation becomes (substituting $\langle \rangle$ for $e^{i\omega t}$):
    \[
    \nabla^2 \Phi = -k^2 \Phi
    \]
Time/space dependancy

- **Characteristic quantities:**
  - angular wave number: $k = \frac{\omega}{c}$
    - analogous to $\omega$ in space, *spatial lineic phase difference*
  - wavelength: $\lambda = \frac{2\pi}{k} = \frac{c(2\pi)}{\omega} = cT$
    - analogous to $T$ in space, *spatial periodicity*

<table>
<thead>
<tr>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$ (rad/s): pulsatation</td>
<td>$k$ (rad/m): wave number</td>
</tr>
<tr>
<td>$f$ (Hz): frequency</td>
<td>$\sigma$ (1/m): répétance (in French)</td>
</tr>
<tr>
<td>$T$ (s): period</td>
<td>$\lambda$ (m): wavelength</td>
</tr>
</tbody>
</table>
## Time/space dependancy

<table>
<thead>
<tr>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$ (rad/s): pulsation</td>
<td>$k$ (rad/m): wave number</td>
</tr>
<tr>
<td>$f$ (Hz): frequency</td>
<td>$\sigma$ (1/m): spatial frequency</td>
</tr>
<tr>
<td>$T$ (s): period</td>
<td>$\lambda$ (m): wavelength</td>
</tr>
</tbody>
</table>
Solutions of the wave equation

- d’Alembert equation (E.A.): complex integration
- general solution
  ➞ specific solutions
  (boundary conditions + initial conditions)

- linearity of E.A.:
  if the specific solutions $p_i$ satisfy E.A., every linear combination is a solution

$$p(t, \vec{r}) = \sum_i p_i(t, \vec{r})$$

➞ superposition principle (justifying Active Noise Control)
Wave structures: plane waves

- definition: one-dimensional propagating wave
  - wave fronts are plans perpendicular to the propagation direction
  - Sound rays are the parallels to the propagation direction

- E.A. (1D): \( \partial_x^2 \nu - \frac{1}{c^2} \partial_t^2 \nu = 0 \) where \( \nu = \Phi, p, s, \xi_x, \) or \( v_x \)

- General solution:
  \[
  \nu(t, x) = \nu_+ (ct - x) + \nu_- (ct + x)
  \]
Plane waves

- **progressive plan wave:**
  
  \( \nu_+(ct-x) \) propagates along increasing \( x \):

  at \((t_1,x_1)\):
  \[ \nu_+(t_1,x_1) \]

  at \((t_2,x_2)\) such as: \( t_2 = t_1 + (x_2-x_1)/c \),
  \[ \nu_+(t_2,x_2) = \nu_+(t_1,x_1) \]
  \( \Rightarrow \) delay \( \Delta t = t_2 - t_1 = (x_2-x_1)/c \)

- \( \nu_+(ct-x) = -\partial_x \Phi_+ = \partial_{(ct-x)} \Phi_+ \)
  \( \Rightarrow \)
  \( p_+(ct-x) = \rho c \partial_{(ct-x)} \Phi_+ = \rho c \nu_+ \)
Wave structures: plane waves
Wave structures: spherical waves

- definition: radial propagating waves
- wave fronts are concentric spheres
- sound rays are half-lines beginning on the centre of the wave fronts (i.e., sphere radius)

- In spherical coordinates:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left[ \cot \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]
\]

→ E.A. only depends on \( r \) and \( t \)
Wave structures: spherical waves
Wave structures: spherical waves

E.A.: \[ \partial_r^2 \Phi + \left(\frac{2}{r}\right) \partial_r \Phi - \partial_t^2 \Phi / c^2 = 0 \]
then: \[ r \partial_r^2 \Phi + 2 \partial_r \Phi - r \partial_t^2 \Phi / c^2 = 0 \]
or \[ \partial_r^2 (r \Phi) = r \partial_r^2 \Phi + 2 \partial_r \Phi \]
\[ \partial_r^2 \beta - \partial_t^2 \beta / c^2 = 0 \]
avec \( \beta = r \Phi \)

this equation is analogous to the one of plan waves:

\[ \Phi(r, t) = \left[ \frac{1}{r} \beta_+ (ct - r) + \frac{1}{r} \beta_- (ct + r) \right] \]
Wave structures: spherical waves

- **Centrifugal spherical wave:**
  \[ \beta_+ \frac{(ct-r)}{r} \]
  propagates along increasing \( r \),
  its amplitude decreases with \( r \)

- **Centripetel spherical wave:**
  \[ \beta_-(ct+r)/r \]
  propagates along decreasing \( r \),
  its amplitude increases \((\to \infty)\)

  this last term then does not have any physical sense

- **Geometrical damping:** decrease of amplitude with the distance \((\text{term } 1/r)\)
Linear acoustics
Sources of sound - radiation
Sources of sound - Radiation

- Hypothesis
  - No disbonding
  - No turbulence (vortex)

Projecting velocity on surface $dS$

$$\vec{v}_0 \cdot \vec{n} = \vec{v} \cdot \vec{n} = v_n$$
Definition: **volume flow velocity**

- Volume flow velocity: it is the volume displaced, by time unit, by the moving « speaker’s » face.
- For the element \( dS \):
  \[
  dq = v_n \cdot dS = \mathbf{v} \cdot \mathbf{n} \cdot dS
  \]
  \( \text{m}^3/\text{s} \)
- For the whole speaker face:
  \[
  q = \int_S v_n \cdot dS
  \]
  \( \text{m}^3/\text{s} \)
Definition: **volume flow velocity**

In the case of the **loudspeaker** (velocity along axis):

\[ \text{velocity } \vec{v}_d \]

\[ dq = \vec{v}_d \cdot \cos \alpha \cdot dS = \vec{v}_d \cdot dS_d \]

\[ q = S_d \cdot \vec{v}_d \quad \text{on a single side} \]

(surface of the equivalent piston)
Sound field of the « pulsating sphere »

- Spherical symmetry of the source
  - spherical wave field

\[
\begin{aligned}
\forall \vec{r}, ||\vec{r}|| \geq a, \\
p(\vec{r}) &= \rho \omega \left( \Phi_1 / r \right) \cdot \exp(-jkr) \\
v(\vec{r}) &= \left[ k - \frac{j}{r} \right] \cdot \left( \Phi_1 / r \right) \cdot \exp(-jkkr)
\end{aligned}
\]

\( \Phi_1 \) can be computed after the fundamental hypothesis of sound radiation
Pulsating sphere

Boundary conditions on the sphere:

\[ v_{|r=a} = \frac{q}{4\pi a^2} \]

\[ v_{|r=a} = \left[ k - \frac{j}{a} \right] \Phi_1 / a \exp(-jka) = \frac{q}{4\pi a^2} \]

then \[ \Phi_1 = \frac{q}{4\pi} \cdot \frac{1}{ka - j} \cdot e^{jka} \]

then:

\[
\begin{aligned}
p(r) &= Z_c k q \frac{\exp[-jk(r-a)]}{4\pi r (ka - j)} \\
v(r) &= \left[ k - \frac{j}{r} \right] q \frac{\exp[-jk(r-a)]}{4\pi r (ka - j)} \\
I(r) &= \frac{\tilde{p}^2}{Z_c} = \frac{Z_c k^2 \tilde{q}^2}{16\pi^2 r^2 [(ka)^2 + 1]} 
\end{aligned}
\]
Pulsating sphere

\[
\begin{align*}
 p(r) &= Z_c k q \frac{\exp[-jk(r-a)]}{4\pi r(ka - j)} \\
 v(r) &= \left[k - \frac{j}{r}\right] q \frac{\exp[-jk(r-a)]}{4\pi r(ka - j)} \\
 I(r) &= \frac{\tilde{p}^2}{Z_c} = \frac{Z_c k^2 \tilde{q}^2}{16\pi^2 r^2 [(ka)^2 + 1]}
\end{align*}
\]

- \(p, v\) and \(I\) independent of \(\theta\) et \(\varphi\)

\[
\begin{align*}
 r &\geq a \\
 r &= |\vec{r}_O - \vec{r}_S| \\
 \vec{r}_O &\Leftrightarrow \vec{r}_S
\end{align*}
\]
Radiation impedance

- Volume flow velocity $q$ induces $p(r)$
- $p(r)$ proportional to $q$

$\Rightarrow$ action/reaction principle

Definition:

$$Z_{ar} = \frac{\left. p \right|_{r=a}}{q} = \frac{Z_c k}{4\pi a (ka - j)}$$

acoustic radiation impedance

$\text{Pa}/(\text{m}^3/\text{s})$, or $\Omega_a$

- force applied on the sphere:
  $$Z_{mr} = \frac{F}{\nu_r}$$
  mécanical rad. imp.

$\text{N}/(\text{m}/\text{s})$, ou $\Omega_m$

$$Z_{mr} = S^2 Z_{ar}$$
Radiation impedance

- **Reminder**: specific acoustic impedance \( Z_s = p/\nu \)
- **Spherical wave**:

\[
Z_s(r) = Z_c \frac{kr}{kr - j}
\]

\[
R_s = Z_c \frac{(kr)^2}{[1 + (kr)^2]}
\]

\[
X_s = Z_c \frac{kr}{[1 + (kr)^2]}
\]
Radiation impedance

- then: \[ Z_{ar} = Z_s(r=a)/S \]
  \[ Z_{mr} = S.Z_s(r=a) \]
- reduced radiation impedance:
  \[ z_r = r_r + j.x_r \]
  \[ = Z_s(r=a)/Z_c \]
  \[ = S.Z_{ar}/Z_c \]
  \[ = Z_{mr}/S.Z_c \]
  \[ \text{with} \quad Z_c = \rho.c \]
Far field of the pulsating sphere

- When \( kr >> 1 \), spherical wavefronts tend to planar wavefronts

- Relationships between \( p \) and \( v \) analogue to plane waves

\[
k r >> 1, \underline{p} \equiv Z_c \underline{v}
\]
Definition: monopole

Ideal punctual source, volume velocity $q$

Field of the monopole, at distance $r$:

$k\alpha \ll 1$

$$p_m = jZ_c q \frac{e^{-jkr}}{4\pi r}$$
Definition: monopole

Ideal ponctual source, volume velocity $q$

Field of the monopole, at distance $r$:

$ka << 1$

$$p_m = jZ_c q \frac{e^{-jkr}}{4\pi r}$$

![Graph showing sound pressure versus distance for $f=100$ Hz.](image)
Complex sources: « vibrating sphere »

Vibrating sphere, with radial movement: $v_r(\theta, \phi)$

General case: no symmetry: computation of $p(r, \theta, \phi)$ after d’Alembert equations

On the contrary from the pulsating sphere, dependancy on $\theta$ and $\phi$

In the following, we consider revolution symmetry

$\Rightarrow$ only dependance on $\theta$
Vibrating sphere

In the general case, \( p(r, \theta) \) can be splitted after:

\[
p(r, \theta) = \sum_{m=0}^{\infty} \alpha_m \cdot P_m(\cos \theta) \cdot h_m^{(2)}(kr)
\]

directivité
dépendance azimuthale
dépendance spatiale

- \( P_0(\cos \theta) = 1 \)
- \( P_1(\cos \theta) = \cos \theta \)
- \( P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1) \)
- \( P_3(\cos \theta) = \frac{1}{2}(5\cos^3 \theta - 3\cos \theta) \)
In the general case, $p(r, \theta)$ can be splitted after:

$$p(r, \theta) = \sum_{m=0}^{\infty} \alpha_m P_m(\cos \theta) h_m^{(2)}(kr)$$

where $h_m^{(2)}(kr)$ are the spherical harmonics and $P_m(\cos \theta)$ are the associated Legendre polynomials.

$$h_0^{(2)}(kr) = j \frac{\exp(-jkr)}{kr}$$

$$h_1^{(2)}(kr) = -\frac{\exp(-jkr)}{kr} \left[ 1 + \frac{1}{jkr} \right]$$

$$h_2^{(2)}(kr) = -j \frac{\exp(-jkr)}{kr} \left[ 1 + \frac{3}{jkr} + \frac{3}{(jkr)^2} \right]$$
We want to:

determine acoustic pressure of the vibrating sphere, after the velocity of the sphere \( v_r(\theta)=v(a).e_r \), and then apply Newton’s law at surface

velocity linked with acoustic pressure for \( r=a \)

Hypothesis: radial movement

de decomposition into "spherical harmonics" (HS) \((HS)\) provides the expression of \( \alpha_m \) determining \( p(r, \theta) \).

\[
v_r(\theta) = \sum_{m=0}^{\infty} v_{rm} P_m (\cos \theta)
\]
Far field

In (HS), it is possible to group terms by (negative) powers of \((kr)\)
\(kr \gg 1\),
\[ p(r, \theta) \approx \exp(-jkr) \]
\(p(r, \theta)\) comprising at least one term in \(1/kr\),
\(\Rightarrow\) its radial component and gradient comprise at least 1 term in \(1/kr\),
And then \(v_r\) comprise at least \(1/kr\) (cf Newton)
\[ \nabla p = \frac{\partial p}{\partial r} e_r + \frac{1}{r} \frac{\partial p}{\partial \theta} e_\theta + \frac{1}{r \sin \phi} \frac{\partial p}{\partial \phi} e_\phi = -\rho \frac{\partial v}{\partial t} \]
though \(v_\theta\) comprise at least terms in \(1/(kr)^2\)
\(\Rightarrow v(r, \theta)\) is radial in far field \((kr \gg 1)\) and \(v \approx v_r = j \frac{\partial p}{\partial r} \frac{1}{kZ_c} \approx \frac{p}{Z_c}\)
\(\Rightarrow\) Same properties as for spherical waves: progressive plane waves in the far field
Directivity

Regrouping by powers of $kr$: 

$$p(r, \theta) \approx Z_c \frac{v_{r0} \exp(-jkr)}{r}$$

$$\Rightarrow p(r, \theta) = D(\theta).p_{m0}$$

Proportionality depends on $\cos(\theta)$ after the (HS): directivity $D(\theta)$

Intensity: 

$$I \cong \tilde{p}^2 / Z_c = D^2(\theta) \underbrace{I_{m0}}_{\text{monopole}}$$

Same properties as pulsating sphere except directivity (magnitude variation along direction)
Radiation of sources

- Can be computed directly in case of simple structures:

\[ Z_{ar} = \frac{p}{q} \]

- Reminder: for the pulsating sphere,

\[ Z_{ar} = \frac{p}{q} = \frac{Z_c k}{4\pi a(ka - j)} \]

- FEM models can ease the computation

- Simple models will be further developed, based on interferences considerations (see next section)
Linear acoustics
Interferences of sound
Properties: interferences

Assuming: - 2 monopoles $M_1$ and $M_2$ ($q_1$ et $q_2$)
- observer distant of $r_1$ and $r_2$ from them

Resulting pressure in O:

\[ p = p_1 + p_2 \]

\[ = p_1 \left[ 1 + \frac{r_1}{r_2} \frac{q_2}{q_1} \exp \left\{ -jk(r_2 - r_1) \right\} \right] \]

\[ = p_1 \left[ 1 + \mu \exp \left\{ -j(k\delta - \phi) \right\} \right] \]
Interferences

- According to $k\delta$, $p$ presents minima and maxima: interferences: difference of phase rotation of $p_1 / p_2$

- The resulting acoustic pressure depends on frequency, but also on the observer’s position.

- If $M_1$ is responsible of the highest pressure amplitude in $O (\mu<1)$:
  
  maxima $(1+\mu)$  
  minima $(1-\mu)$
Spatial dependancy

- Equal-phase locations:
  \[ k \delta - \Phi = \Theta + 2n\pi, \]
  \( \Theta \) being constant and \( n \) a positive integer

Then \( \delta = (\Theta/2\pi + \Phi/2\pi + n) \lambda = (\theta + \phi + n) \lambda \)

This ensemble is a revolution hyperboloïd whose focal points are \( M_1 \) and \( M_2 \) (or hyperboles in plan comprising \( M_1 \) and \( M_2 \)).

Pressure minima: \( \delta = \{n + 1/2 + \phi\} \lambda \)

Pressure maxima: \( \delta = \{n + \phi\} \lambda \)
Frequential dependancy

Hypothesis: $O(x_o, y_o)$, out of median plan of $M_1M_2$

- pressure presents a succession of min and max depending on frequency: « filtrage en peigne »

- Particular case: 2 monopoles in phase ($\Phi=0$)
  - 1\textsuperscript{st} minimum for $f_c = c/2\lambda$ (fréquence de peigne)
  - 1\textsuperscript{st} maximum for $2f_c$
  - Alternate of min and max separated by $f_c$

If $\Phi \neq 0$, shift of frequency: $f_\Phi = \Phi c/2\pi\lambda$
Examples

- In sound diffusion: 2 identical loudspeakers, placed at each side of the scene, restituting a same signal

- Resulting acoustic pressure vs. frequency: for an observer O, different distance from the 2 loudspeakers
Example

The same applies for **stereophonic sound recording**
Comments

- Interferences play an important role in audio
- When suffered: try to limitate their influence
- When created: microphone and loudspeakers arrays

- Active control: profits the interferences to reduce noise
  - It is aimed at creating, within spatial areas, destructive interferences with one or many sources (loudspeakers),
  - The command of sources (electric signal) is computed to create sound fields opposing to the primary sound field
Energy

\[ \tilde{\rho} = \sqrt{\left( \sum_{m=1}^{i} p_m(t) \right)^2} = \sqrt{\left( \sum_{m=1}^{i} \tilde{p}_m^2(t) \right) + \sum_{m,n}^{m \neq n} p_m(t) p_n(t)} \]

energetical dependency
Correlation coefficient

Energetical dependancy between 2 waves is illustrated by the correlation coefficient

\[ \rho_{mn} = \rho_{nm} = \frac{\langle p_m p_n \rangle}{\tilde{p}_m \tilde{p}_n} \]

If \( \rho_{mn} = 1 \), \( p_m \) and \( p_n \) are correlated.

On the contrary, if \( p_m \) and \( p_n \) are independant:

\( \rho_{mn} \) is null.
Correlation: pure tones (same freq.)

Objective: find $\rho_{12}$?

$$\langle p_1 p_2 \rangle = 2 \text{Re} \left[ p_1 \exp(j\omega t) \right] \text{Re} \left[ p_2 \exp(j\omega t) \right]$$

Denoting:

$$\tilde{p}_2 = \tilde{p}_1 \left( \tilde{p}_2 / \tilde{p}_1 \right) \exp(j\phi_{21})$$

Then the average over one period is:

$$\rho_{12} = \cos \phi_{21}$$
Correlation: pure tones (same freq.)

Discussion:

\[ \phi_{21} = 2n \pi \]
\[ \phi_{21} = (2n+1) \pi \]
\[ \phi_{21} = (2n+1/2) \pi \]

Energetical sum: if \( \rho_{mn} = 0 \), energetical sum is allowed
Definitions

• A harmonic sound is composed of a superposition of harmonics (pure tones) of a single frequency (=fundamental).
• The fundamental frequency is the least divider of the series of frequencies.
• An harmonic series is composed with harmonic sounds which fundamentals are integer-multiples of the lower fundamental of the series.
• An inharmonic sound is composed of a superposition of pure tones which frequencies do not define integer intervals.
Properties

- For one harmonic:
  
  \[ p_m(t) = \sqrt{2} \text{Re}\left\{ \tilde{p}_m \exp(jm2\pi ft) \right\} \]

  \( m = \text{rank of the harmonic} \)

  \( f = \text{fundamental frequency} \)

Then:

\[ p(t) = \sum_{m=1}^{i} p_m(t) = \sqrt{2} \text{Re}\left\{ \sum_{m=1}^{i} \tilde{p}_m \exp(jm2\pi ft) \right\} \]

where \( i \) is the max value of \( m \) up to which \( \tilde{p}_m \) is not null.

Orthogonality: \( \rho_{mn} = 0 \)
Linear acoustics
Refraction of sound
Planar interface

boundary conditions:
1) no resulting force between media
2) no disbonding, no turbulence: always in contact

hypothesis:
- losseless media,
- harmonic waves,
- interfaces, obstacles > λ,
- locally plan waves
Definitions

• at the interface (x=0):
  
  reflection coefficient:
  \[ r = \frac{p_r}{p_i} \]
  
  transmission coefficient:
  \[ t = \frac{p_t}{p_i} \]

• Issue:

  find \( r, t \), and \( \theta_2 \)

  knowing \( Z_{c1}, Z_{c2}, c_1, c_2 \) and \( \theta_1 \)
Relationships on acoustic pressure

- plan waves in any direction

\[ p_+(\vec{r}) = p_{+0} \exp(-jk \cdot \vec{r}) = p_{+0} \exp(-jk_x x - jk_y y - jk_z z) \]

\[ \vec{k}(k_x, k_y, k_z) = k \vec{n}_+ \] (directing cosinus)

we get

\[
\begin{align*}
\vec{p}_i &= \vec{p}_i \exp[-jk_1 (x \cos \theta_1 + y \sin \theta_1)] \\
\vec{p}_r &= \vec{r} \cdot \vec{p}_i \exp[-jk_1 (-x \cos \theta_1 + y \sin \theta_1)]
\end{align*}
\]

since \( \vec{k}_1 = [k_1 \cos \theta_1 ; k_1 \sin \theta_1] \) et \( k_1 = \frac{\omega}{c_1} \)

\[
\begin{align*}
\vec{p}_t &= t \cdot \vec{p}_i \exp[-jk_2 (x \cos \theta_2 + y \sin \theta_2)] \\
since \vec{k}_2[k_2 \cos \theta_2 ; k_2 \sin \theta_2] \text{ et } k_2 = \frac{\omega}{c_2}
\end{align*}
\]
Boundary conditions

- the 1st boundary condition leads to:
  \[ p_i(x=0) + p_r(x=0) = p_t(x=0) \]

since

\[ 1 + r = t \exp[-j y (k_2 \sin \theta_2 - k_1 \sin \theta_1)] \]

then

\[ k_2 \sin \theta_2 - k_1 \sin \theta_1 = 0 \]

and

\[ 1 + r = t \]

for all \( y \)
Boundary conditions

- the 2\textsuperscript{nd} boundary condition leads to the equality of normal components of acoustic velocities (1 and 2)

\[ \vec{v}_i \cdot \cos \theta_i - \vec{v}_r \cdot \cos \theta_i = \vec{v}_t \cdot \cos \theta_2 \]

then

\[ \left( \frac{p_i}{Z_1} \right) \cdot \cos \theta_1 - r \cdot \left( \frac{p_i}{Z_1} \right) \cdot \cos \theta_1 = t \cdot \left( \frac{p_i}{Z_2} \right) \cdot \cos \theta_2 \]

\[ \Rightarrow (1 - r) \cdot Z_2 \cdot \cos \theta_1 = Z_1 \cdot t \cdot \cos \theta_2 \]
Properties

- **Snell’s law**

  \[ k_2 \sin \theta_2 - k_1 \sin \theta_1 = 0 \ \Rightarrow \ \frac{\sin \theta_1}{\sin \theta_2} = \frac{c_1}{c_2} = n_{12} \] (refraction index)

  Values of \( r \) and \( t \) (after boundary conditions)

  \[ r = r = \frac{Z_2 \cdot \cos \theta_1 - Z_1 \cdot \cos \theta_2}{Z_2 \cdot \cos \theta_1 + Z_1 \cdot \cos \theta_2} \]

  \[ t = t = \frac{2 \cdot Z_2 \cdot \cos \theta_1}{Z_2 \cdot \cos \theta_1 + Z_1 \cdot \cos \theta_2} \]

  - \( r > 0 \)  \( p_i \) and \( p_r \) in phase, \( \nu_i \) and \( \nu_r \) phase opposites
  - \( r \leq 0 \)  \( p_i \) and \( p_r \) phase opposites, \( \nu_i \) et \( \nu_r \) in phase
  - \( t \) always \( > 0 \)  \( p_i \) and \( p_t \) in phase (always!)
Definitions

- Reflection factor:

\[ \rho = \frac{I_r}{I_i} = \frac{\tilde{p}_r^2}{\tilde{p}_i^2} = r^2 \]

- Transmission factor:

\[ \tau = \frac{I_t}{I_i} = \frac{\tilde{p}_t^2}{\tilde{p}_i^2} = \frac{4Z_1Z_2 \cos^2 \theta_1}{(Z_2 \cos \theta_1 + Z_1 \cos \theta_2)^2} \]
Definitions

\[ \rho = \frac{P_{ar}}{P_{ai}} \]
\[ \tau \neq \frac{P_{at}}{P_{ai}} \]

d’où \( \tau' = \frac{P_{at}}{P_{ai}} = \left( \frac{S_2}{S_1} \right) \cdot \tau = 1 - \rho \)
(transmission factor in terms of power)

Reflection and transmission levels

\[ L_\rho = 10 \log \rho \]
\[ L_\tau = 10 \log \tau' \]
Particular case: total reflection

- $c_1 < c_2 \Rightarrow \theta_2 > \theta_1$ (from low to high celerity medium)
- Limit angle
  \[
  \theta_1 = \arcsin \left( \frac{c_1}{c_2} \right)
  \]
  \[
  \theta_2 = \frac{\pi}{2}
  \]
- Evanescent wave in the vicinity of the interface

- $L_p = 0 \text{ dB}, L_r = -\infty \text{ dB}$
- If we remind the expression:

  \[
  r = r \cdot \exp(j\phi)
  \]

  \[
  r = 1 \quad p_r = p_i
  \]

  $\phi = \ldots$ discontinuity
Particular case: total transmission

- $r=0$ for $Z_2 \cdot \cos \theta_1 - Z_1 \cdot \cos \theta_2 = 0$

- Intromission or Brewster’s angle verifies this condition

$$\theta_{1B} = \arcsin \left\{ \left( \frac{\rho_2}{\rho_1} \right)^2 - n_{12}^2 \right\}^{1/2}$$

- It is real if
  - $l \leq n_{12} \leq \rho_2 / \rho_1$ for $\rho_2 / \rho_1 \geq l$
  - $\rho_2 / \rho_1 \leq n_{12} \leq l$ for $\rho_2 / \rho_1 \leq l$
Particular case: total transmission

Transparency conditions:

\[ Z_1 = Z_2 \text{ et } c_1 = c_2 \]

then \[ \rho_1 = \rho_2 \text{ et } c_1 = c_2 \]

Example: hydrophone
Particular case: normal incidence

\[ \theta_1 = \theta_2 = 0 \rightarrow r = \frac{Z_2 - Z_1}{Z_2 + Z_1} \]

\[ \tau = \tau' = 1 - \rho = 1 - r^2 \]

matching:

\[ Z_1 = Z_2 \rightarrow r = 0 \rightarrow \tau = 1 \]

Remarque: si

\[ Z_1 = 2 \cdot Z_2 \text{ ou } 1/2 \cdot Z_2 \rightarrow r = 0.3 \rightarrow \tau = 0.9 \]

\[ L_\tau = -0.5 \text{ dB} \]

adaptation condition is not too demanding
Media with losses

- introduce \( \vec{\gamma} = \vec{a} + jk \)
- Analogy with planar discontinuity for EM plan wave
- important case: only medium 2 presents losses

\[
\begin{align*}
Z_1 & | Z_2 = R_2 + jX \\
k_1 & | \vec{\gamma}_2 = \vec{a}_2 + jk_2 \\
c_1 & | \text{non-uniform} \\
& | \text{transmitted wave}
\end{align*}
\]
Media with losses

- results: 
  \[ r = \frac{Z_2 \cos \theta_1 - Z_1 \cos \theta_2}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2} \]
  \[ \rho = |r|^2 < 1 \text{ no more total reflection} \]

- transmitted wave

  \[ p_{t} = \tilde{p}_i (1 + r) \cdot \exp\left( k_1 (\sin^2 \theta_1 \cdot n_{12}^2)^{1/2} x \right) \cdot \exp(-jk_1 \sin \theta_1 y) \]

  with \( n_{12} = \left( \frac{c_1}{c_2} \right) \left[ 1 - j \frac{a_2}{k_2} \right] \)

  \[ c_2 = \phi_2 = \frac{\omega}{k_2} \]
Fluid-solid interface

- much more complex since both longitudinal and transverse (shearing) waves occur in solids
- oblique incidence → 2 wave types
- normal incidence → only longitudinal waves!

former results still valid!!!
Fluid-solid interface

- locally reacting solid

  of high importance in practice

  simple,

  often verified

At each point of the solid, the movement is normal and independant of the other points
Characteristic impedance of a material

\[ Z_n = \frac{p}{v_n} = R_n + jX_n \text{ Ns/m}^3 \]

Localised reaction: \( Z_n \) constant on the interface, independant of \( \theta_i \) but in general: \( Z_n(f) \)

Properties: 2 boundary conditions

\[ r = \frac{(Z_n \cos \theta_1 - Z_1)}{(Z_n \cos \theta_1 + Z_1)} \]

then

\[ \rho = |r|^2 \]

\[ \tau = \tau' = 1 - \rho = \frac{4R_nZ_1 \cos \theta_1}{\left( R_n \cos \theta_1 + Z_1 \right)^2 + X_n^2} \]
Example

**Ground effect**
interferences between a primary wave and its reflection on an interface

1 source + 1 **image source**
the volume flow (of image source) depends on the 2 media properties

**General case:** uneasy, depends on incidence angle
Example

Ground effect

Particular case: razing incidence

⇒ reflection coefficient tends to -1

(Reflection + change of sign)

⇒ phase opposition between incident and reflected waves

⇒ about the same magnitude

Then: decrease of $p$ is in $1/r^2$ and not $1/r$ anymore
Linear acoustics
guided waves in 1D
Formal analogies

Analogies between:

- acoustics
- electromagnetism
- electrotechnics

⇒ Analogous mathematical models:

\textit{formal analogies}
Lossless transmission lines

- Bifilar line

\[ \partial_x U = - j\omega L' I \]

\[ \partial_x I = - j\omega C' U \]
Plan acoustic wave

- Direct analogy

\[
\frac{\partial}{\partial x} p = -j \omega \rho v
\]
\[
\frac{\partial}{\partial x} v = -j \omega K^{-1} p
\]
Plan acoustic wave

• Inverse analogy

\[ \partial_x v = - j \omega K^{-1} p \]
\[ \partial_x p = - j \omega \rho v \]
Reciprocity principle

Let’s assume a medium submitted to 2 acoustic fields

\[ p_1, v_1 \]
\[ p_2, v_2 \]

As in electromagnetics, we come to the following:

\[ \int_{S} (v_2 \cdot \underline{p}_1 - v_1 \cdot \underline{p}_2).dS = 0 \]

\[ \rightarrow \text{reciprocity principle} \]
Solution of a O.P. problem

Harmonic plan wave

\[ p(x) = p_+(x) + p_-(x) = p_{+0} \exp(-j k x) + p_{-0} \exp(+ j k x) \]
\[ v(x) = v_+(x) + v_-(x) = v_{+0} \exp(-j k x) + v_{-0} \exp(+ j k x) \]

and

\[ p_{+0} = Z_c v_{+0} \]
\[ p_{-0} = -Z_c v_{-0} \]

there remains only 2 boundary conditions to solve the problem…
Plan mode ducts

Duct: - section $S$
  - rectilign and uniform
  - lateral walls are rigid and smooth

Compressible and lossless fluid

Plane mode: only plane waves are propagating

\[ q(x) = S \mathcal{V}(x) \]

\[ p_{\pm} = \pm Z_{ac} q_{\pm} \]

\[ Z_a(x) = \frac{p(x)}{q(x)} \]
Analogies with lossless transmission lines

\[
\begin{align*}
\frac{\partial}{\partial x} p(x) &= -j\omega \left( \frac{\rho}{S} \right) q(x) \\
\frac{\partial}{\partial x} q(x) &= -j\omega \left( \frac{S}{K} \right) p(x)
\end{align*}
\]

Reflection coefficient \( r(x) \)
Acoustic impedance \( Z_a(x) \)
Stationary wave ratio \( s \)
Transmitted acoustic power ➔ transmission factor \( \tau \)
Boundary conditions

Let’s assume a source at the entrance of the duct:

- oscillating piston,
- volume flow $q_0$ (ideal volume flow source)

At the piston ($x=0$):

$$ q_0 = q_{+0} + q_{-0} $$
Input impedance

\[ Z_{a0} = Z_a (x = 0) \]
\[ = Z_{ac} [Z_{al} + jZ_{ac} \tan(kl)] / [Z_{ac} + jZ_{al} \tan(kl)] \]
Example

Duct with a **rigid** output (ie null flow velocity):

\[
\begin{align*}
q(x = 0) &= q_0 = q_{+0} + q_{-0} \\
q(x = l) &= q_{+0} \exp(-jkl) + q_{-0} \exp(+jkl) = 0
\end{align*}
\]

we deduce

\[
\begin{align*}
q_{+0} &= q_0 \exp(+jkl) / 2 j \sin kl = 0 \\
q_{-0} &= -q_0 \exp(+jkl) / 2 j \sin kl = 0
\end{align*}
\]
Example

Duct with a **rigid** output (ie null flow velocity):

\[
\begin{align*}
q(x = 0) &= q_0 = q_{+0} + q_{-0} \\
q(x = l) &= q_{+0} \exp(-jkl) + q_{-0} \exp(+jkl) = 0
\end{align*}
\]

we deduce

\[
q(x) = q_{+0} \exp(-j k x) + q_{-0} \exp(+j k x) \\
= q_0 \left\{ \exp[+ j k (l - x)] - \exp[- j k (l - x)] \right\} / 2 j \sin kl \\
= q_0 \sin k (l - x) / \sin kl \\
p(x) = Z_{ac} q_{+0} \exp(-j k x) - Z_{ac} q_{-0} \exp(+j k x) \\
= - j Z_{ac} q_0 \cos k (l - x) / \sin kl
\]
Example

Duct with an open output (ie null resulting pressure):

\[
\begin{align*}
q(x = 0) &= q_0 = q_\theta + q_0 \\
p(x = l) &= p_{+0} \exp(-jkl) + p_{-0} \exp(+jkl) = 0
\end{align*}
\]

we deduce

\[
\begin{align*}
q(x) &= q_\theta \cos(kl) + \chi \cos(kl) \\
p(x) &= jZ_{ac} q_\theta \sin(kl) + \chi \sin(kl)
\end{align*}
\]

Input impedances:

\[
\begin{align*}
d\text{closed} & \quad Z_{a0} = Z_{ac} x = \infty jZ_{ac} kl \\
d\text{open} & \quad Z_{a0} = jZ_{ac} kl
\end{align*}
\]
Application: Kundt tube

Standard device for measuring the specific acoustic impedance of the material at the end of the duct

Acoustical characterization of materials (as used in the building industry)