Coating by Cubes

K. Bezdek\(^1\) and T. Hausel\(^1\)

Department of Geometry, Eötvös L. University
1088 Budapest, Rákóczi út 5, Hungary

1. Introduction

Let \( P_0, P_1, \ldots, P_n \) be convex \( d \)-polytopes in \( d \)-dimensional Euclidean space with pairwise disjoint interiors. We say that \( P_0 \) is \emph{coated} by \( P_1, \ldots, P_n \) if \( P_0 \subseteq \text{int} \left( \bigcup_{i=0}^{n} P_i \right) \), where \( \text{int}(\cdot) \) stands for the interior of the corresponding set. Coating occurs very often in a very natural way. For example, in each tiling every tile is coated by its neighbors. Thus, if we take an arbitrary triangulation of \( E^d \), then the number of neighbors of any tile is at least as large as the minimum number of \( d \)-simplices that can coat a \( d \)-simplex in \( E^d \). In this connection the following problem is a rather very basic question.

**Problem 1.** Find the minimum number of \( d \)-simplices that can coat a \( d \)-simplex in \( E^d \).

The answer to the above question is obviously three in \( E^2 \). In general, we know only the following.

**Proposition.** Every \( d \)-simplex can be coated by \((2d - 1)\) \( d \)-simplices in \( E^d \), where \( d \geq 2 \).

Since the number of facets of a \( d \)-cube in \( E^d \) is \( 2d \), the number of \( d \)-cubes that can coat a fixed \( d \)-cube is at least \( 2d \). The following theorem formulates a sharper statement under some conditions.

**Theorem.** Let \( P_0 \) be a \( d \)-cube of edgelength \( \lambda \) with edges parallel to the coordinate-axes of \( E^d \). Moreover, let \( P_1, \ldots, P_n \) be a collection of unit \( d \)-cubes with edges parallel to the coordinate-axes of \( E^d \) such that \( P_0 \) is coated by \( P_1, \ldots, P_n \).

1. If \( 0 < \lambda < 1 \), then \( n \geq 2^d \), where equality can be achieved for any \( 0 < \lambda < 1 \) and \( d \geq 1 \).
2. If \( \lambda = k \) is a positive integer, then \( n \geq 2(k + 1)^d - 2k^d \), where equality can be achieved for any \( k \geq 1 \) and \( d \geq 1 \).

\(^1\) The work was partially supported by the Hung. Nat. Foundation for Sci. Research number 326-0413.
As a result we get the following:

**Corollary.** The minimum number of the translates of a $d$–cube that can coat a given $d$–cube in $E^d$ is at least $2^d$, where $d \geq 1$. If all $d$–cubes are translates of each other, then $2^d$ can be replaced by $2^{d+1} - 2$.

**Problem 2.** Prove or disprove that the minimum number of $d$–cubes that can coat a $d$–cube in $E^d$ is $2^{d-1} + 2$, where $d \geq 2$.

2. **Proof of the Proposition**

We prove the statement by induction on the dimension $d$. As the claim is obviously true for $d = 2$ we may assume that it is true for any $d' < d$ with $d \geq 3$. Thus, let $S$ be a $d$–simplex in $E^d$ with vertices $v_1, v_2, \ldots, v_{d+1}$. Moreover, let $H$ be the hyperplane in $E^d$ spanned by the vertices $v_1, v_2, \ldots, v_d$ and let $S_0$ be the $(d-1)$–simplex with vertices $v_1, v_2, \ldots, v_d$. By induction there are $(d-1)$–simplices $S_1, S_2, \ldots, S_{2d-3}$ that coat $S_0$ in $H$. Let $v$ be a point in $E^d$ such that $v_{d+1}$ is the relative interior point of the segment $v_1v$ and let $v'$ be a point in $E^d$ that is strictly separated from $v$ by $H$. Then it is easy to see that the $d$–simplices $\text{conv}(S_1 \cup \{v\}), \text{conv}(S_2 \cup \{v\}), \ldots, \text{conv}(S_{2d-3} \cup \{v\}), \text{conv}\{v_2, v_3, \ldots, v_{d+1}, v\}$ and $\text{conv}(S_0 \cup \{v'\})$ coat the $d$–simplex $S$, where $S_0$ is a simplex in $H$ containing $S_0$ in its relative interior. This completes the proof of the Proposition.

3. **Proof of the Theorem**

**Proof of (1).** In the following proof we assume only that the edgelengths of the $d$–cubes $P_1, \ldots, P_n$ are larger than $\lambda$.

At first, remove the $d$–cubes of the collection $P_1, \ldots, P_n$ that are disjoint from $P_0$. Let $P_1, \ldots, P_n$ denote the system left. Obviously, $P_1, \ldots, P_n$ still coat $P_0$. We are going to show that $n = 2^d$. Recall that an orthant in $E^d$ is the closure of a connected component of the complement of $d$ pairwise orthogonal hyperplanes of $E^d$.

**Lemma 1.** Each $d$–cube $P_i, 1 \leq i \leq n$ can be replaced by an orthant $O_i$ with $P_i \subset O_i$ such that the edges of the orthants $O_1, \ldots, O_n$ are parallel to the coordinate–axes of $E^d$ and the interiors of the orthants $O_1, \ldots, O_n$ are pairwise disjoint.

**Proof.** Take a $d$ cube $P_i, 1 \leq i \leq n$. Let $v_i$ be the vertex of $P_i$ that lies closest to the $d$–cube $P_0$. Then let $O_i$ be the orthant with apex $v_i$ and with edges parallel to the coordinate–axes of $E^d$ and with $P_i \subset O_i$. We are going to show that each $O_i$ is disjoint from the interiors of the $d$–cubes $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$ and then we prove that the interiors of the orthants $O_1, \ldots, O_n$ are pairwise disjoint indeed. In order to do so we need the following:

**Lemma 2.** Let $H$ be the hyperplane of any facet of $P_i$ that does not contain $v_i$. Then $H \cap P_0 = \emptyset$.

**Proof.** (Indirect) Assume that $H \cap P_0 \neq \emptyset$. Then take the orthogonal projection of $v_i$ onto $H$. This is a vertex say $v_i'$ of $P_i$. Moreover, let $w_i$ be the point of $P_0$ that is closest to $v_i$.
and let $w'_i$ be the orthogonal projection of $w_i$ onto $H$. Obviously, as $H \cap P_0 \neq \emptyset$ we have $w'_i \in P_0$. Finally, as the edgelength of $P_0$ is smaller than the edgelength of $P_i$ we get that $\text{dist}(v_i, w_i) > \text{dist}(v'_i, w'_i)$. Thus, $\text{dist}(v_i, P_0) > \text{dist}(v'_i, P_0)$, a contradiction.

Now imagine a $d$–cube $P_j$, $j \neq i$ with int $P_j \cap \text{int } O_i \neq \emptyset$. Recall that int $P_j \cap \text{int } O_i = \emptyset$. Then obviously, there exists a facet of $P_i$ the hyperplane $H$ of which separates int $P_j$ from int $P_i$. As int $P_j \cap \text{int } O_i \neq \emptyset$ therefore $v_i \notin H$. Hence, Lemma 2 implies that $H \cap P_0 = \emptyset$. Now, recall that $P_i \cap P_0 \neq \emptyset$ and $P_j \cap P_0 \neq \emptyset$. Consequently, $H$ (that separates $P_i$ from $P_j$) must intersect (the convex set) $P_0$, a contradiction. Hence, we proved that int $O_i \cap \text{int } P_j = \emptyset$ for any $i \neq j \in \{1, \ldots, n\}$. In order to finish the proof of Lemma 1 we proceed as follows. Take $O_1$ and enlarge $P_1$ from $v_1$ by a very large factor obtaining the cube $P'_1$ the vertex $v_1$ of which is still the closest vertex to $P_0$. As a result of the previous arguments $P'_1, P_2, \ldots, P_n$ coat $P_0$. Then enlarge $P_2, P_3, \ldots, P_n$ after each other in order to get a coating system of $P_0$ using rather large $d$–cubes. Keep doing this to see that the orthants $O_1, \ldots, O_n$ have pairwise disjoint interiors. This completes the proof of Lemma 1.

Apply Lemma 1 to get a system $\{O_1, \ldots, O_n\}$ of orthants with $P_i \subset O_i$ and with edges parallel to the coordinate–axes of $\mathbb{E}^d$ such that the orthants $O_1, \ldots, O_n$ have pairwise disjoint interiors, where $1 \leq i \leq n$. Obviously, no two of the orthants $O_1, \ldots, O_n$ are translates of each other and they coat $P_0$. Thus, $n \leq 2^d$. Take the $2^d - 1$ orthants with edges parallel to the coordinate–axes of $\mathbb{E}^d$ that share the same vertex of $P_0$ as an apex and are disjoint from the interior of $P_0$. Then it is easy to see that any $O_i$ intersects the interior of $2^d$ orthants out of $2^d - 1$. Hence, there must be an $O_i$ that intersects the interior of one orthant out of $2^d - 1$ implying that its apex $v_i$ is a vertex of $P_0$. Thus, $n \geq 2^d$ and so $n = 2^d$.

We are left with the proof of showing the existence of $2^d$ orthants $O_1, \ldots, O_{2^d}$ that coat $P_0$. As a result of the previous arguments we look for $2^d$ orthants with the property that the apex of each orthant is a vertex of $P_0$ and each vertex of $P_0$ is an apex of exactly one orthant. We prove the existence of such orthants by induction on the dimension $d$. They obviously exist in case $d = 2$. So assume that if $P'_0$ is a $(d-1)$–cube of edgelength $\lambda$ with edges parallel to the coordinate–axes of $\mathbb{E}^{d-1}$, then there are $2^{d-1}$ orthants in $\mathbb{E}^{d-1}$ say, $O'_1, \ldots, O'_{2^{d-1}}$ that coat $P'_0$ in $\mathbb{E}^{d-1}$. Also, assume that $\mathbb{E}^{d-1}$ is a hyperplane of $\mathbb{E}^d$. Then for each orthant $O'_i$, $1 \leq i \leq 2^{d-1}$ in $\mathbb{E}^{d-1}$ we assign two orthants of $\mathbb{E}^d$ say, $+O_i$ and $-O_i$ such that the distinct orthants $+O_i$ and $-O_i$ share the $(d-1)$–dimensional orthant $O'_i$ as a facet in common. Let $F'_1$ and $F'_2$ be two opposite (i.e., disjoint) facets of $P'_0$. Without loss of generality we may assume that the apexes of the orthants $O'_1, \ldots, O'_{2^{d-2}}$ belong to $F'_1$ and the apexes of the orthants $O'_{2^{d-2}+1}, \ldots, O'_{2^d-1}$ belong to $F'_2$. Finally, let $e_1$ be the vector of length $\lambda$ with $e_1 + F'_1 = F'_2$ and let $e_d$ be a vector of length $\lambda$ orthogonal to $\mathbb{E}^{d-1}$. Without loss of generality we may assume that the $d$–dimensional orthants $+O_1, \ldots, +O_{2^{d-1}}$ lie in that closed half–space of $\mathbb{E}^d$ bounded by the $\mathbb{E}^{d-1}$ into which $e_d$ points.

Then take the following $2^d$ orthants in $\mathbb{E}^d$:

$$
\begin{align*}
&\epsilon_d + (-O_1), \epsilon_d + (-O_2), \ldots, \epsilon_d + (-O_{2^{d-2}}); \\
&\epsilon_1 + \epsilon_d + (+O_1), \epsilon_1 + \epsilon_d + (+O_2), \ldots, \epsilon_1 + \epsilon_d + (+O_{2^{d-2}}); \\
&-\epsilon_1 + (-O_{2^{d-2}+1}), -\epsilon_1 + (-O_{2^{d-2}+2}), \ldots, -\epsilon_1 + (-O_{2^{d-1}});
\end{align*}
$$
If $P_0$ is the $d$-cube $\text{conv}(P'_0 \cup (e_d + P'_0))$, then using the induction hypothesis that $P'_0$ is coated by the $(d-1)$-dimensional orthants $O'_1, \ldots, O'_{2^{d-1}}$ in $E^{d-1}$ it is easy to see that the above $2^d$ $d$-dimensional orthants coat $P_0$ in $E^d$. This completes the proof of (1). \qed

Proof of (2). At first, we show that if $P_0$ has an integer edgelength say $k \geq 1$, then $P_0$ can be coated by $2(k+1)^d - 2k^d$ unit $d$-cubes with edges parallel to the coordinate axes of $E^d$, i.e., parallel to the edges of $P_0$. We prove this by induction on the dimension $d$. The claim is obviously true for the case $d = 1$. So assume that it is true for every $d' < d$ and take a $d$-cube $P_0$ of $E^d$ with integer edgelength $k \geq 1$. Let $H_0$ be a supporting hyperplane of $P_0$ that intersects $P_0$ in a facet $F_0$. Then let $H_l$ be the translate of $H_0$ by the vector of length $l$ orthogonal to $H_0$ that intersects $P_0$ in a $(d-1)$-cube $F_l = H_l \cap P_0$ of edgelength $k$, where $l = 1, \ldots, k$. By induction each $F_l$ can be coated by $2(k+1)^{d-1} - 2k^{d-1}$ unit $(d-1)$-cubes in $H_l$, where $l = 0, 1, \ldots, k$. Thus, if we place $k(2(k+1)^{d-1} - 2k^{d-1})$ unit $d$-cubes between the consecutive hyperplanes $H_i, H_{i+1}$, $0 \leq i \leq k-1$ properly, then we are left with the problem to coat $P_0$ along the facets $F_0$ and $F_k$ only. This can be done easily by $2(k+1)^{d-1}$ unit $d$-cubes. Thus, $P_0$ is coated by $k(2(k+1)^{d-1} - 2k^{d-1}) + 2(k+1)^{d-1} = 2(k+1)^d - 2k^d$ unit $d$-cubes in $E^d$ finishing the construction.

At second, notice that each unit $d$-cube of the above construction has a $(d-1)$-dimensional intersection with $P_0$. The following easy lemma is the key to prove the claim (2) completely.

**Lemma 3.** Let $P_0$ be a $d$-cube of $E^d$ with integer vertices and with edges parallel to the coordinate-axes of $E^d$. We assign to $P_0$ each orthant of $E^d$ that has an integer apex belonging to $P_0$ and the edges of which are parallel to the coordinate-axes of $E^d$ such that the interior of the orthant is disjoint from $P_0$. If $P$ is a unit $d$-cube of $E^d$ with edges parallel to the coordinate-axes of $E^d$ such that $P \cap P_0$ is $(d-1)$-dimensional, then the number of the orthants assigned to $P_0$, each of whose interior intersects $P$ and each of whose apex belongs to $P$, is always $2^{d-1}$.\[4pt\]

**Proof.** We leave the rather easy proof to the reader. \[\]

To complete the proof of (2) assume that $P_0$ is coated by the unit $d$-cubes $P_1, \ldots, P_n$ in $E^d$. Without loss of generality we may assume that $P_0$ is a $d$-cube of $E^d$ with integer vertices and with edges parallel to the coordinate-axes of $E^d$. Assign to $P_0$ all orthants described in Lemma 3. As $P_i \cap P_0$ is at most $(d-1)$-dimensional ($1 \leq i \leq n$) it is easy to see (using Lemma 3) that the interior of each of which intersects $P_i$ and the apex of each of which belongs to $P_i$ is at most $2^{k-1}$. Thus, a very simple counting argument implies that $n$ is at least as large as the number of unit $d$-cubes in the above construction, i.e., $2(k+1)^d - 2k^d$. This completes the proof of (2). \[\]

### 4. Proof of the Corollary

Without loss of generality we may assume that $P_0$ is a $d$-cube of edge length $\lambda$ with edges parallel to the coordinate-axes of $E^d$ such that it is coated by the unit $d$-cubes $P_1, \ldots, P_n$.
of $E^d$ the edges of which are parallel to the coordinate-axes of $E^d$. If $\lambda \leq 1$, then the claim follows from the Theorem in a straight way. So, we are left with the case, when $\lambda > 1$. Then take two opposite facets say, $F$ and $F'$ of $P_0$. Obviously, $F$ ($F'$, resp.) is a $(d-1)$-cube of edge length $\lambda$ that is covered by some $d$-cubes of the collection $P_1, \ldots, P_n$ each of which has edge length smaller than $\lambda$. Thus, the number of $d$-cubes of the collection $P_1, \ldots, P_n$ that cover $F$ ($F'$, resp.) is obviously at least $2^{d-1}$. Hence, $n \geq 2^{d-1} + 2^{d-1} = 2^d$. 

**Acknowledgement.** We are indebted to the referee for the valuable remarks which made the presentation more clear.

Received 14.10.93