Kac’s conjectures on quiver representations via arithmetic harmonic analysis

Topology of Hitchin map and arithmetic of character variety

based on joint work with M. de Cataldo and L. Migliorini

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Diffeomorphic spaces in non-Abelian Hodge theory

- $C$ genus $g$ curve; fix group $\text{GL}_n$

$$\mathcal{M}^d_{\text{Dol}} := \left\{ \text{moduli space of semistable rank } n \text{ degree } d \text{ G-Higgs bundles } (E, \phi) \right\}$$  
  i.e. $E$ rank $n$ degree $d$ bundle on $C$, $\phi \in H^0(C, \text{ad}(E) \otimes K)$ Higgs field

$$\mathcal{M}^d_{\text{DR}} := \left\{ \text{moduli space of flat } \text{GL}_n\text{-connections on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} \text{id around } p \right\}$$

$$\mathcal{M}^d_{\text{B}} := \{A_1, B_1, \ldots, A_g, B_g \in G | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} \text{id} \} \mod G$$

When $(d, n) = 1$ these are smooth non-compact varieties

**Theorem (Non-Abelian Hodge Theorem)**

$$\mathcal{M}^d_{\text{Dol}} \overset{\text{diff}}{=} \mathcal{M}^d_{\text{DR}} \overset{\text{diff}}{=} \mathcal{M}^d_{\text{B}}$$
(Deligne 1972) proved the existence of 
\[ W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q}) \] for any complex algebraic variety \( X \), which is

- functorial
- compatible with cup-product

(Hausel-Villegas 2008) calculates

\[
E(\mathcal{M}_B; q) = |\mathcal{M}_B(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n)
\]

we find \( E(\mathcal{M}_B; 1/q) = q^d E(\mathcal{M}_B; q) \) palindromic by Alvis-Curtis duality

\[
q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))
\]

\( \leadsto \) Curious Hard Lefschetz Conjecture (theorem when \( n = 2 \)):

\[
L^l : \text{Gr}^W_{d-2l}(H^{i-l}(\mathcal{M}_B)) \to \text{Gr}^W_{d+2l}H^{i+l}(\mathcal{M}_B) \quad \chi \mapsto \chi \cup \alpha^l,
\]

where \( \alpha \in W_4 H^2(\mathcal{M}_B) \)

The implied functional equation on the conjectured \( H(\mathcal{M}_B; q, t) = (qt)^d n H(\mathcal{M}_B; \frac{1}{qt^2}, t) \) holds
Perverse filtration

- $f : X \to Y$ a \textit{proper} map between complex algebraic varieties of relative dimension $d$

- (de Cataldo-Migliorini 2005) introduce \textit{perverse filtration}
\[ P_i \subset P_{i+1} \subset \ldots P_k(X) \cong H^k(X) \] from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves

- the Relative Hard Lefschetz Theorem holds:
\[
L^l : \text{Gr}^{P}_{d-l}(H^*(X)) \to \text{Gr}^{P}_{d+l}H^{*+2l}(X)
\]
\[ x \mapsto x \cup \alpha^l \]

where $\alpha \in H^2(X)$ is a relative ample class
Main conjecture

- recall Hitchin map

\[ \chi : \mathcal{M}_{\text{Dol}} \rightarrow \mathbb{A} \cong \bigoplus_{i=1}^{n} H^0(C; K^i) \]

\[ (E, \phi) \mapsto \text{charpol}(\phi) \]

(Hitchin 1987) → completely integrable Hamiltonian system and proper


\[ P_k(\mathcal{M}_{\text{Dol}}) \cong W_{2k}(\mathcal{M}_B) \text{ under the isomorphism} \]

\[ H^*(\mathcal{M}_{\text{Dol}}) \cong H^*(\mathcal{M}_B) \text{ from non-Abelian Hodge theory} \]

- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when \( X \) smooth and \( Y \) affine:
  take \( Y_0 \subset \cdots \subset Y_i \subset \cdots Y_d = Y \)
  s.t. \( Y_i \) generic with \( \dim(Y_i) = i \) then

\[ P_{k-i-1} H^k(X) = \ker(H^k(X) \rightarrow H^k(f^{-1}(Y_i))) \]

- thus Conjecture ⇒ "topology of Hitchin map reflects the arithmetic of the character variety"
now on let \( n = 2 \), i.e. study GL(2) Higgs bundles

\( \mathcal{E} \rightarrow \mathcal{M}_{\text{Dol}} \times \Sigma \) and \( \Phi : \mathcal{E} \rightarrow \mathcal{E}K \), universal Higgs bundle

\((\mathcal{E}, \Phi) |_{(E,\phi) \times \Sigma} = (E, \phi)\)

\[
c_2(\text{End}(\mathcal{E})) = 2\alpha [\Sigma]^* + \sum_{i=1}^{2g} 4\psi_i e_i - \beta
\]

for some \( \alpha \in H^2(\mathcal{M}) \), \( \psi_i \in H^3(\mathcal{M}) \) and \( \beta \in H^4(\mathcal{M}) \).

Generate \( H^*(\mathcal{M}_{\text{Dol}}(\text{PGL}_2)) \). (Hausel-Thaddeus 2004)

(Hausel-Villegas 2008) \( \Rightarrow \alpha, \psi_i, \beta \in W_4 \),

Conjecture \( \Rightarrow \alpha, \psi_i, \beta \in P_2 \Rightarrow \psi_i, \beta \in \ker(H^*(\mathcal{M}_{\text{Dol}}) \rightarrow H^*(\chi^{-1}(Y_0)))\)

Yes! was proved by (Thaddeus 1990)

\( \beta \in P_2 H^4(\mathcal{M}_{\text{Dol}}) \) would mean

\( \beta \in \ker(H^4(\mathcal{M}_{\text{Dol}}) \rightarrow H^4(\chi^{-1}(Y_1))) \) i.e. \( \beta \) vanishes over a generic curve in \( \mathbb{A} \).
Applications of Ngô’s support theorem

**Theorem (Ngô, 2008)**

\[ \chi^{\text{ell}} : \mathcal{M}_{\text{ell}} \subset \mathcal{M}_{\text{Dol}} \rightarrow \mathbb{A}_{\text{ell}} \subset \mathbb{A} \text{ over points with integral spectral curve.} \]

\[ R\chi^{\text{ell}}_* \mathbb{Q} \cong \bigoplus_{i \geq 0} IC_{\mathbb{A}_{\text{ell}}}(L^{\wedge i})[-i], \]

where \( L^{\wedge i} = R^i\chi^{\text{ell}}_*(\mathbb{Q}) = \wedge^i R^1\chi^{\text{ell}}_*(\mathbb{Q}) \) on \( \mathbb{A}_{\text{reg}} \), where spectral curve is smooth.

Applications for \( n = 2 \):

- \((2 - 2g)\beta = c_2(\mathcal{M}_{\text{Dol}})\) vanishes on \( \mathbb{A}_{\text{reg}} \). Ngô’s support theorem \( \Rightarrow \beta \) vanishes on the generic line \( \Rightarrow \beta \in P_2 \).
- By computation \( IC_{\mathbb{A}_{\text{ell}}}(L^{\wedge i}) = j^\text{reg}_*(L^{\wedge i}) \) , i.e. no higher cohomology sheaves \( \Rightarrow \) perverse filtration on \( H^*(\mathcal{M}_{\text{Dol}}^{\text{ell}}) \) is compatible with cup-product \( \Rightarrow P \subset W \) on \( H^*(\mathcal{M}_{\text{Dol}}) \) (almost) \( \Rightarrow \) CHL (Thm for \( n = 2 \)) and RHL imply \( P = W \) (almost)
One more ingredient needed: the intersection form $H^*_c(M_{Dol}) \to H^*(M_{Dol})$ is trivial when $n = 2$ by (Hausel, 1998)

Thus in particular $\beta^i \in P_{2i}(H^{4i}(M_{Dol}))$

i.e. vanishes over a generic $2i - 1$ dimensional subvariety in $M_{Dol}$

the pure subring $\langle 1, \beta, \ldots, \beta^{g-1} \rangle$ is dual by RHL with the $g$-dimensional $H^{mid}(M_{Dol})$

thus $\dim P_{mid/2-2i}/P_{mid/2-2i-1}H^{mid}(M_{Dol}) = 1$ for $i = 0, 1, \ldots, g - 1$ and 0 otherwise

consequently $\sum_i q^i \dim P_{mid/2-i}H^{mid}(M_{Dol}) = A_{S_g}(2, q)$;

where $S_g$ is the $g$-loop quiver
A-polynomial and perverse filtration

- C genus $g$ Riemann surface with punctures $a_1, \ldots, a_k \in \mathbb{C}$
- $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}(n)^{1..k}$
- $\mathcal{M}^\mu_{\text{Dol}}$ moduli space of stable parabolic Higgs bundles with generic weights at the quasi-parabolic structure at $a_i$ of type $\mu^i$
- For every $\mu$ and $g$ one can find generic weights $\sim \mathcal{M}^\mu_{\text{Dol}}$ is always smooth
- Can arrange that $\mathcal{M}^\mu_{\text{Dol}} \overset{\text{diff}}{\simeq} \mathcal{M}^\mu_{\text{B}}$

**Conjecture**

$$P_k(\mathcal{M}^\mu_{\text{Dol}}) \overset{\sim}{=} W_{2k}(\mathcal{M}^\mu_{\text{B}}) \text{ under the isomorphism}$$

$$H^*(\mathcal{M}^\mu_{\text{Dol}}) \overset{\sim}{=} H^*(\mathcal{M}^\mu_{\text{B}})$$

**Corollary**

$$\sum_i q^i \dim P_{\text{mid}/2-i} H^\text{mid} \left( \mathcal{M}^\mu_{\text{Dol}} \right) = A_{\Gamma_\mu} (\alpha_\mu, q) \text{ in particular}$$

$$\dim H^\text{mid} \left( \mathcal{M}^\mu_{\text{Dol}} \right) = A_{\Gamma_\mu} (\alpha_\mu, 1) \text{ and}$$

$$\dim \text{Im}(H^\text{mid}_c \left( \mathcal{M}^\mu_{\text{Dol}} \right)) = A_{\Gamma_\mu} (\alpha_\mu, 0) = m_{\alpha_\mu} \text{ for}$$

$$\text{Im}(H^\text{mid}_c \left( \mathcal{M}^\mu_{\text{Dol}} \right)) \subset H^\text{mid} \left( \mathcal{M}^\mu_{\text{Dol}} \right)$$
Hilbert schemes of points on surfaces

Let $C = E$ elliptic curve, $k = 1$ and $\mu = (\mu^1)$ with
$\mu^1 = (n - 1, 1)$

Then one can show that $H^*(M^\mu_B) \cong H^*((\mathbb{C}^\times \times \mathbb{C}^\times)^[n])$

preserving the weight filtration

One can also show that $M^\mu_{\text{Dol}} \cong (T^*E)^[n]$ and the Hitchin map
is just $(T^*E)^[n] \rightarrow (T^*E)^{(n)} \rightarrow \mathbb{C}^{(n)}$

Theorem (de Cataldo, Hausel, Migliorini 2009)

$P_k(H^*((T^*E)^[n])) \cong W_{2k}(H^*((\mathbb{C}^\times \times \mathbb{C}^\times)^[n]))$ under the canonical
isomorphism $H^*((T^*E)^[n])) \cong H^*((\mathbb{C}^\times \times \mathbb{C}^\times)^[n])$

in this case $\mu$ is indivisible and the quiver variety
$M_{\alpha_\mu} \cong (\mathbb{C}^2)^[n]$

$A_{\Gamma_\mu}(\alpha_\mu, q) = q^n P((\mathbb{C}^2)^[n], 1/\sqrt{q})$ by Crawley-Boevey-Van den
Bergh

so in this case
$\sum_i q^i \dim P_{\text{mid}/2-i}H^{2n}((T^*E)^[n]) = q^n P((\mathbb{C}^2)^[n], 1/\sqrt{q}) = A_{\Gamma_\mu}(\alpha_\mu, q)$
Cohomology of quiver varieties $\leadsto$ representation theory of Kac-Moody algebras

their Poincaré polynomials $\leadsto$ A-polynomials of quiver representations

cohomology of character varieties and moduli of parabolic Higgs bundles (with extra filtrations) $\leadsto$ deformation of cohomology of quiver varieties

their weight polynomials $\leadsto$ deformations of A-polynomials given by Macdonald polynomials

What is the corresponding deformation of the Kac-Moody algebra?

Can consider $\text{SL}_n$ and $\text{PGL}_n$ instead of $\text{GL}_n$ $\leadsto$

  Hausel-Thaddeus mirror symmetry conjecture & Ngô’s proof of the fundamental lemma in the Langlands program
  $\leadsto$ p-adic harmonic analysis

Connection with our arithmetic harmonic analysis?