Kac’s conjectures on quiver representations via arithmetic harmonic analysis

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Plan for three lectures

Lecture 1: Representations of quivers; Kac’s conjectures; Betti numbers of Nakajima’s quiver varieties; proof of Kac’s conjecture 1

Lecture 2: Cohomology of character and quiver varieties; attack on Kac Conjecture 2

Lecture 3: Topology of Hitchin map and arithmetic of character variety; another attack on Kac Conjecture 2
a quiver $\Gamma$ is an oriented and connected graph with vertices $I = (1, \ldots, n)$ and arrows or oriented edges $E \subset I \times I$, (possibly multiple edges and edge-loops)

denote $a = (t(a), h(a)) \in E$ the tail and head of the arrow $a$

$K$ field; (either $\mathbb{C}$ or $\mathbb{F}_q$)

a representation $\rho$ of $\Gamma$ is a collection of finite dimensional $K$-vector spaces $\{V_i\}_{i \in I}$ and homomorphisms $\rho_a \in \text{Hom}(V_{t(a)}, V_{h(a)})$ for every $a \in E$

$\text{dim } \rho = (\text{dim } V_1, \ldots, \text{dim } V_n) \in \mathbb{N}^I$ is the dimension of $\rho$

Example: Let $S_g$ be the quiver on one vertex and $g$ loops. Classifying representations of $S_g$ of dimension $(d)$ is classifying the isomorphism classes of $g$ tuples of $d \times d$ matrices. Representations of $S_1$ (matrices up to conjugation) are classified by Jordan normal form.
The $A$-polynomial

- $\alpha \in \mathbb{N}^l$ a dimension vector
- A quiver representation is *absolutely indecomposable* if it cannot be written as a direct sum of non-trivial quiver representations over $\overline{K}$

$$A_\Gamma(\alpha, q) := \left\{ \text{abs. indec. reps of } \Gamma \text{ over } \mathbb{F}_q \text{ of dimension } \alpha, \text{ modulo isomorphism} \right\}$$

**Theorem (Kac, 1982)**

$A_\Gamma(\alpha, q) \in \mathbb{Z}[q]$ and is independent of the orientation of $\Gamma$. 
Kac’s conjectures

**Conjecture (Kac, 1982)**

1. When $\Gamma$ is loopless, the constant term $A_\Gamma(\alpha, 0) = m_\alpha$
2. $A_\Gamma(\alpha, q) \in \mathbb{N}[q]$, i.e. the coefficients of $A_\Gamma(\alpha, q)$ are $\geq 0$.

Both conjectures were known to Kac for finite and affine quivers and for the ”polygon”-quiver

**Theorem (Crawley-Boevey, Van den Bergh 2004)**

*Both conjectures hold true for any quiver with $\alpha$ indivisible; i.e. $\gcd(\alpha(i)) = 1$*

Every quiver supports infinitely many divisible dimension vectors $\sim$ both conjectures remained open for any wild quiver.

Here we prove Conjecture 1 in Lecture 1 and explain two attacks on Conjecture 2 for comet-shaped quivers in Lectures 2 & 3.
Weight filtration

- $X$ variety defined over $\mathbb{Z}$
- Jordan decomposition of $\text{Frob}_q$ on $H^k_c(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \Rightarrow$ weight filtration $W_l \subset H^k_c$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2} i \leq l$
  ($\Leftarrow$ Weil’s Riemann hypothesis $\Leftarrow$ Deligne 1974)
- comparison theorem: $H^*_c(X(\mathbb{C}); \mathbb{C}) \cong H^*_c(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \otimes \mathbb{C}$
- (Deligne 1971) constructs weight filtration on $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_k = H^k_c(X(\mathbb{C}); \mathbb{Q})$ which is functorial when $W_{k-1} \cap H^k_c(X; \mathbb{Q}) = 0$
  the weight filtration is pure;
  e.g. when $X$ is smooth projective;
  or when $X \subset \overline{X}$, with $\overline{X}$ smooth projective and injects on $H^*_c$
  e.g. when $X$ is a symplectic quiver variety; a Nakajima quiver variety, $\mathcal{M}_{\text{DR}}$ moduli space of flat connections and $\mathcal{M}_{\text{Dol}}$ the moduli space of Higgs bundles on a Riemann surface
- weight filtration is not pure or mixed e.g. for $X = \text{GL}_n$ or for $\mathcal{M}_B$ the character variety of representations of the fundamental group of a Riemann surface to $\text{GL}_n$
Arithmetic and topological content of the E-polynomial

- For a complex variety $X$ define $E$-polynomial
  
  $E(X; q) = \sum \dim(W_i/W_{i-1}(H^k_c(X)))(-1)^k q^{i/2}$

- additive - if $X_i \subset X$ locally closed s. t. $\cup X_i = X$ then
  
  $E(X; q) = \sum E(X_i; q)$

- multiplicative - $F \to E \to B$ locally trivial in the Zariski topology
  
  $E(E; q) = E(B; q)E(F; q)$

- when weight filtration is pure
  
  $E(X; q) = \sum \dim(H^k_c(X))(-q^{1/2})^k$ is the Poincaré polynomial

- if all eigenvalues $\lambda_i$ of Frob$_q$ on $H^*_c(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell)$ are integer powers of $q$, then $|X(\mathbb{F}_{q^n})| = \sum \lambda_i^n$ is a polynomial in $q^n$ and

  $= E(X; q)$

**Theorem (Katz 2006)**

*If $X$ is a variety defined over $\mathbb{Z}$ and $\#\{X(\mathbb{F}_q)\} = E(q)$ is a polynomial in $q$, then $E(M; q) = E(q)$.*

- e.g. if $E(q) \in \mathbb{Q}[q] \implies E(q) \in \mathbb{Z}[q]$ proves Kac’s result that

  $A_{\Gamma}(\alpha, q) = \#\{Z(\Gamma, \alpha)(\mathbb{F}_q)\} \in \mathbb{Z}[q]$

  $Z(\Gamma, \alpha)$ is variety parametrizing $\Gamma$-indecomposables of dim $\alpha$
Linear symplectic quotients

- $G$ complex reductive group; $V$ finite dimensional complex vector space
- Assume $G$ acts on $V$ linearly via the representation $\rho : G \to \text{GL}(V)$, with derivative $\varrho : g \to \text{gl}(V)$
- Symplectic structure on $M := V \times V^*$ given by $\omega((v_1, w_1), (v_2, w_2)) = w_1(v_2) - w_2(v_1)$
- $G$ acts on $V \times V^*$ symplectically via the representation $\rho \oplus \rho^*$ where $\rho^* : G \to \text{GL}(V^*)$ is the dual representation
- This action is Hamiltonian with moment map $\mu : V \times V^* \to g^*$
  $$\mu(v, w)(X) = \langle \varrho(X)v, w \rangle$$
- For $\xi \in (g^*)^G$ we have the linear symplectic quotient $M///\xi G = \mu^{-1}(\xi)//G$
For a quiver $\Gamma$ and dimension vector $\alpha$ let $\{ V_i \}_{i \in I}$ be a collection of finite dimensional vector spaces of dimension $\alpha$

- $\mathbb{V}_\alpha = \bigoplus_{a \in E} \text{Hom}(V_{t(a)}, V_{h(a)})$
- $G_\alpha = \prod_{i \in I} \text{GL}(V_i) / \text{GL}_1$, where $\text{GL}_1 = (\lambda, \ldots, \lambda)_{\lambda \in \text{GL}_1} < \prod_{i \in I} Z(\text{GL}(V_i)) < \prod_{i \in I} \text{GL}(V_i)$
- its Lie algebra $g_\alpha = \{ X_i \in \text{gl}(V_i) | \sum_i \text{tr}(X_i) = 0 \} \subset \prod_i \text{gl}(V_i)$
- action $\rho : G_\alpha \to \text{GL}(\mathbb{V}_\alpha)$ from left and right
- for a generic $\xi \in (g_\alpha^*)^{G_\alpha}$ define the quiver variety by

$$M_\alpha = \mathbb{V}_\alpha \times \mathbb{V}_\alpha^* \sslash \sslash \xi G_\alpha$$

- if $\alpha \in \mathbb{N}^I$ is indivisible (gcd($\alpha$) = 1) then $M_\alpha$ is non-singular, while if $\alpha$ is divisible (gcd($\alpha$) > 1) $M_\alpha$ has singular points (when non-empty).
- when non-empty dim $M_\alpha = 2 - 2(\alpha, \alpha)$
- (Crawley-Boevey, Van den Bergh 2004) when $\alpha$ indivisible

$$|M_\alpha(\mathbb{F}_q)| = q^{1-(\alpha, \alpha)} A_\Gamma(\alpha, q) \& H_c^*(M_\alpha; \mathbb{Q})$$

is pure $\leadsto$ $q^{1-(\alpha, \alpha)} A_\Gamma(\alpha, q) = P_c(M_\alpha, q^{1/2}) \in \mathbb{N}[q]$

$\leadsto$ Kac’s Conjecture 2 when $\alpha$ indivisible
Nakajima quiver varieties

• \( \mathbf{v}, \mathbf{w} \in \mathbb{N}^I \) and \( \dim(V_i) = v_i \) and \( \dim(W_i) = w_i \) then
  \[ G_{\mathbf{v}} = \times_{i \in I} \text{GL}(V_i) \]
  naturally acts on
  \[ \forall_{\mathbf{v}, \mathbf{w}} = \bigoplus_{(i, j) \in E} \text{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \text{Hom}(W_i, V_i) \]
  the corresponding holomorphic symplectic quotient
  \[ \mathcal{M}(\mathbf{v}, \mathbf{w}) = \mu_{\mathbf{v}, \mathbf{w}}^{-1}(\mathbf{1}_\mathbf{v})/\!/G_{\mathbf{v}} \]

is the affine Nakajima quiver variety

• always non-singular of dimension
  \[ 2d_{\mathbf{v}, \mathbf{w}} = 2\left(\sum_{(i, j) \in E} v_i v_j + \sum_{i \in I} v_i (w_i - v_i)\right) \]

• Crawley-Boevey’s trick: to a quiver \( \Gamma \) with two dimension vectors \( \mathbf{v}, \mathbf{w} \in \mathbb{N}^I \) \( \sim \) \( \Gamma_w \) which has \( n + 1 \) vertices \( I' = \{1, \ldots, n, \ast\} \) with the same oriented arrows on \( I \subset I' \) and \( w_i \) arrows from \( \ast \) to \( i \). Then one can identify
  \[ \mathcal{M}(\mathbf{v}, \mathbf{w}) = \mathcal{M}^{\Gamma_w}_{(\mathbf{v}, 1)}, (\mathbf{v}, 1) \text{ is clearly indivisible } \Rightarrow \]
  \[ P_c(\mathcal{M}_{\mathbf{v}, \mathbf{w}}; q^{1/2}) = q^{d_{\mathbf{v}, \mathbf{w}}} A_{\Gamma_w}((\mathbf{v}, 1), q) \]
Fourier transform on $g^*$

- $V$ finite dimensional vector space over $\mathbb{F}_q$
- $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ non-trivial additive character
- $f : V \to \mathbb{C}$ its Fourier transform $\hat{f} : V^* \to \mathbb{C}$

$$\hat{f}(Y) := \sum_{X \in V} f(X)\Psi(\langle X, Y \rangle).$$

- Fourier inversion formula (FIF): $\hat{\hat{f}}(X) = |V|f(-X)$
- Recall $G$ acts on $V$, with derivative $\varrho : g \to gl(V)$, inducing action on $M := V \times V^*$, Hamiltonian with moment map $\mu : M \to g^*$, given by $\mu(v, w)(X) = \langle \varrho(X)v, w \rangle$
- For $\xi \in g^*(\mathbb{F}_q)$ the count function of the moment map

$$\#_\mu : g^*(\mathbb{F}_q) \to \mathbb{N} \subset \mathbb{C}$$

$$\#_\mu(\xi) := \#\{(v, w) \in M(\mathbb{F}_q)|\mu(v, w) = \xi\} = \sum_{(v, w) \in M} \delta_{\mu(v, w)}(\xi)$$

**Proposition (Hausel, 2006)**

$$\hat{\#}_\mu(x) = |V||\ker\varrho(x)| \xrightarrow{FIF} \#_\mu = \frac{|V|}{|g|} \hat{a}_\varrho,$$

where $a_\varrho(x) = |\ker\varrho(x)|$
Theorem (Hausel 2006)

For any quiver $\Gamma$, and $w \in \mathbb{N}^I$ the Betti numbers of Nakajima quiver varieties are:

$$\sum_{v \in \mathbb{N}^I} \sum_i \dim(H^2_i(M(v, w))) q^{i - d(v, w)} X^v =$$

$$= \sum_{v \in \mathbb{N}^I} \sum_{\lambda \in \mathcal{P}(v)} \left( \prod_{(i, j) \in E} q^{\langle \lambda^i, \lambda^j \rangle} \right) \left( \prod_{i \in I} q^{\langle \lambda^i, (w^i) \rangle} \right) \left( \prod_{i \in I} q^{\langle \lambda^i, (w^i) \rangle} \right) \prod_{k} \prod_{j=1}^{m_k} (1 - q^{-j})$$

where $2d(v, w) = 2 \sum_{(i, j) \in E} v_i v_j + 2 \sum_{i \in I} v_i (w_i - v_i)$ is the dimension of $M(v, w)$, $X^v = \prod_{i \in I} T_i^v$ and $\langle \lambda, \mu \rangle = \sum_{i, j} \min(\lambda_i, \mu_j)$.
Theorem (Kac 1974)

Let \( L(w) \) be an irreducible representation of \( g(\Gamma) \) of highest weight \( \Lambda \in P \). Let \( L(\Lambda) = \bigoplus_{\alpha \in \mathbb{N}^I} L(\Lambda)_{\Lambda - \alpha} \) denote its weight space decomposition. Then

\[
\sum_{\alpha \in \mathbb{N}^I} \dim (L(\Lambda)_{\Lambda - \alpha}) X^\alpha = \sum_{w \in W} \det(w) X^{\Lambda + \rho - w(\Lambda + \rho)} \prod_{\alpha \in \mathbb{N}^I} (1 - X^\alpha)^{m_\alpha}
\]

Theorem (Nakajima 1998)

Fix \( w \in \mathbb{N}^I \) then there is an irreducible representation of the Kac-Moody algebra \( g(\Gamma) \) of highest weight \( \Lambda_w \) on

\[
\bigoplus_{v \in \mathbb{N}^I} H^2_{d_v,w}(M(v,w)), \text{ in particular }\]

\[
\sum_{v \in \mathbb{N}^I} \dim \left( H^2_{d_v,w}(M(v,w)) \right) X^v = \sum_{w \in W} \det(w) X^{\Lambda_w + \rho - w(\Lambda_w + \rho)} \prod_{\alpha \in \mathbb{N}^I} (1 - X^\alpha)^{m_\alpha}
\]
Proof of Kac’s Conjecture 1

Weyl-Kac-Nakajima formula + our main formula \( \leadsto \)

\[
\sum_{w \in W} \det(w) X^{\Lambda_w + \rho - w(\Lambda_w + \rho)} = \prod_{\alpha \in \mathbb{N}^I} (1 - X^{\alpha})^{m_\alpha} = \left( \sum_{v \in \mathbb{N}^I} X^v \sum_{\lambda \in \mathcal{P}(v)} \frac{\prod_{(i,j) \in E} q^{\langle \lambda^i, \lambda^j \rangle}}{\prod_{i \in I} (q^{\langle \lambda^i, \lambda^i \rangle} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1 - q^{-j}))} \right)_{q=0}
\]

when \( w = m1 \), i.e. \( \Lambda_w = m\rho \), \( m \to \infty \) and \( A_\Gamma(\alpha, q) = \sum_i t_i^\alpha q^i \)

\[
\prod_{\alpha \in \mathbb{N}^I} (1 - X^{\alpha})^{m_\alpha} = \left( \sum_{v \in \mathbb{N}^I} X^v \sum_{\lambda \in \mathcal{P}(v)} \frac{\prod_{(i,j) \in E} q^{\langle \lambda^i, \lambda^j \rangle}}{\prod_{i \in I} (q^{\langle \lambda^i, \lambda^i \rangle} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1 - q^{-j}))} \right)_{q=0}
\]

\[
Hua \left( \prod_{\alpha \in \mathbb{N}^n} \prod_{j=0}^\infty \prod_{i=0}^\infty (1 - q^{i+j} X^{\alpha})^{t_i^\alpha} \right)_{q=0} = \prod_{\alpha \in \mathbb{N}^n} (1 - X^{\alpha})^{t_i^\alpha}
\]

**Theorem (Hausel 2006)**

\( A_\Gamma(\alpha, 0) = m_\alpha \)