Betti numbers of large hyperkähler manifolds

joint work with Fernando R. Villegas

Tamás Hausel

Chair of Geometry, EPF Lausanne
http://geom.epfl.ch/Hausel/talks/pdf

Algebraic geometry seminar
Universität Duisburg-Essen, June 2013
How geometry effects topology?

- conjecture of (Hopf, 1930') $\rightsquigarrow$
  compact Riemannian $M^{2m}$ such that $\sec(M) > 0 \Rightarrow \chi(M) > 0$

- $M^{2m}$ compact Kähler manifold $\Rightarrow$ odd Betti numbers even
  (e.g. Hopf surface $\cong S^3 \times S^1$ has no Kähler metric)

- $M^{2m}$ Kähler $\Leftrightarrow$ holonomy $\subset U(m)$ $\rightsquigarrow$ $\omega \in \Omega^2(M)$

- Hard Lefschetz (HLT):
  $$L^l : H^{m-l}(M) \cong H^{m+l}(M)$$
  $$x \mapsto x \cup [\omega]^l$$

- smooth projective toric variety $M^{2m}$ Kähler
  HLT gives all constraints on Betti numbers (Stanley 1980) $\rightsquigarrow$
  classification of face vectors of simple rational convex polytopes
Hyperkähler manifolds

- $M^{4m}$ hyperkähler when holonomy $\subset \text{Sp}(m) \iff$ Kähler wrt complex structures $I, J, K$ satisfying $IJK = -1$
- there are few known compact examples; many non-compact but complete examples:
  - toric hyperkähler manifolds $\mathcal{M}_H$; e.g. toric quiver varieties $\mathcal{M}_1^\Gamma$
  - Hilbert scheme of $n$-points on $\mathbb{C}^2$: $(\mathbb{C}^2)^[n]$
  - moduli spaces of YM instantons on $\mathbb{R}^4 \sim$ ADHM spaces $\mathcal{M}_{n,m}$
  - Nakajima quiver varieties $\mathcal{M}_v^\Gamma$
  - moduli space of Higgs bundles on a Riemann surface $\mathcal{M}_n^g$
- all of these examples are semiprojective

$\iff \mathbb{C}^\times \not\subset \mathcal{M}_I$ s.t. $\lim_{\lambda \to 0} \lambda z$ exists for all $z \in M$

$\text{core } C := \{x \in M | \lim_{\lambda \to \infty} \lambda z \text{ exists}\}$ projective
Betti numbers of semiprojective hyperkähler manifolds

- (Elingsrud–Strømme 1987) \( b_{2i}((\mathbb{C}^2)^n) = \#\{\text{partitions of } n \text{ into } i \text{ parts}\} \)

- (Bielawski–Dancer 2000) \( b_{2i}(\mathcal{M}_H) = h_i(H) \) h-number of hyperplane arrangement

- (Hausel–Sturmfels 2002) \( P_t(\mathcal{M}_\Gamma) = \sum b_{2i}(\mathcal{M}_\Gamma^\Gamma) t^{2i} = \frac{t^{2d}}{(1-t^2)^{n-1}} \text{Rel}_\Gamma(1/t^2) \) reliability polynomial of \( \Gamma \)

- (Crawley-Boevey–Van den Bergh 2004) \( P_t(\mathcal{M}_\Gamma^v) = t^{2dv} A_v^\Gamma(1/t^2) \) Kac polynomial counting absolutely indecomposable representations of quiver \( \Gamma \) of dimension \( v \)

- (Hausel–Villegas 2008) \( P_t(\mathcal{M}_n^g) \) conjecture for \( P_t(\mathcal{M}_n^g) \)

- (Chuan–Diaconescu–Pan 2012) \( \leftrightarrow \) our conjecture \( \leftrightarrow \) refined Gopakumar-Vafa conjecture

\( \leftrightarrow \) arithmetic harmonic analysis on finite Lie groups, algebras

all the formulas are combinatorially tractable
semiprojective hyperkähler manifold $M^{4m} \sim C$
$\sim b_i(M^{4m}) = 0$ when $i > \dim C = 2m$

(Hausel 2006) $\sim$ semiprojective hyperkähler manifold satisfies weak Hard Lefschetz:

$$L : H^i(M^{4m}) \leftrightarrow H^{i+2}(M^{4m})$$

$$x \mapsto x \cup [\omega_i]$$

for $i \leq \min \dim(C)/2 = m$

(Hausel-Sturmfels 2002)

$\sim$ restrictions on Betti numbers of toric hyperkähler varieties
$\sim$ new restrictions on face vectors of matroid complexes

weak Hard Lefschetz for $M^g_n$ also follows from $P = W$ conjecture of (de Cataldo–Hausel–Migliorini 2010)

What do Poincaré polynomials of semi-projective hyperkähler varieties look like?
Gaussian distribution as limit

- Gaussian distribution: \( \exp\left(-\frac{x^2}{2}\right) \)
e.g. distribution of chest circumference in scottish army 1846
- central limit theorem (CLT) \( \sim \) appropriately rescaled, the graphs of the polynomials \( P_t(M^g_1) = (1 + t)^{2g} \) tend to the Gaussian distribution

\[
P_t(T^* \text{Gr}(n, k)) = \begin{bmatrix} n \\ k \end{bmatrix}_{t^2} = \prod_{i=1}^{k} \frac{1 - q^{n+1-i}}{1 - q^i}
\]

\[
P_t(T^* \text{Gr}(n, 1)) = \begin{bmatrix} n \\ 1 \end{bmatrix}_{t^2} \rightarrow 1 \text{ as } n \rightarrow \infty;
\]

\[
P_t(T^* \text{Gr}(n, k)) = \begin{bmatrix} n \\ k \end{bmatrix}_{t^2} \rightarrow 1^k \text{ as } n \rightarrow \infty;
\]

CLT \( \sim 1^k \rightarrow \text{Gaussian as } k \rightarrow \infty \)

- conjecture: “\( \lim_{m \to \infty} \lim_{n \to \infty} P_t(M_{n,m}) \) is Gaussian

(Morrison 2013) \( \sim \) appropriately rescaled

“\( P_t((\mathbb{C}^3)[n]) = DT_n(\mathbb{C}^3; q) \) tend again to Gaussian as \( n \to \infty \)
Gumbel distribution as limit

- Gumbel distribution: $e^x e^{-e^x}$
  - e.g. distribution of maximum annual value of daily rainfalls
- (Ellingsrud–Strømme 1987) $\leadsto$
  - $b_{2i}((\mathbb{C}^2)[n]) = \#\{\text{partitions of } n \text{ into } i \text{ parts}\}$
- (Erdős–Lehner, 1941) $\leadsto$ the distribution of
  - $\#\{\text{partitions of } n \text{ into } i \text{ parts}\}$ is governed by the Gumbel distribution
- i.e. $P_t((\mathbb{C}^2)[n])$ tends to Gumbel as $n \to \infty$
- $\#\{\text{partitions of } n \text{ into } i \text{ parts}\} =$
  - $\#\{\text{partitions of } n \text{ with largest part } i\}$
Airy distribution as limit

- Airy distribution: implicitly given by its (complicated) moments e.g. area under Brownian motion
- (Hausel–Sturmfels, 2002) \( P_t(M_{\Gamma}^\Gamma) = t^{2d} A_{\Gamma}^\Gamma(1/t^2) \)
- when \( \Gamma = K_n \) the complete graph on \( n \) vertices

\[
A_{\Gamma}^\Gamma(1 + q) = \sum_k c_{n,k} q^k,
\]

\( c_{n,k} = \# \{ \text{connected graphs on } n \text{ labelled vertices of genus } k \} \) are the moments of the discrete distribution given by \( A_{\Gamma}^\Gamma \)

- asymptotics determined by (Wright 1977) \( \sim \) \( \lim_{n \to \infty} P_t(M_{K_n}^{1}) \) approaches the Airy distribution

- (Reineke, 2005) computed \( P_t(H_{n,1}^{(g)}) \) for certain non-commutative Hilbert schemes \( H_{n,1}^{(g)} \)
- limiting behaviour of \( P_t(H_{n,1}^{(g)}) \) is Airy as \( n \to \infty \)
- expect \( P_t(M_{\Gamma}^\Gamma) \to \text{Airy as } \mathbf{v} \to \infty \) in many/most directions
Questions

- Is there a limiting geometrical object \( \lim_{n \to \infty} M_n \) in our examples?

- Does it have a ”continuously graded” cohomology, with Poincaré distribution given by the observed limiting distribution?

- Is \( A^\Gamma_1 \) convergent for every large graph limit sequence \( \Gamma_n \) of Lovász –Szegedy?

- large \( N \)'t Hooft limit is \( N \to \infty \) in certain \( U(N) \) gauge theories, where string theory appears. Is any of our limiting observations relevant to this limit?

- \( \lim_{n \to \infty} P_t(M^g_n) =? \)

- \( \lim_{n \to \infty} H_{t,q}(M^g_n) =? \)