Topology of the Hitchin map
and the arithmetic of the character variety
Project in progress with Mark De Cataldo - Luca Migliorini

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Hard Lefschetz Theorem

- Hodge theorem for compact oriented Riemannian manifold $M$: $H^*(M) \cong \mathcal{H}^*(M)$ the space of $L^2$ harmonic forms
- If $M$ is additionally Kähler we have the action of an $N = 2$ supersymmetry algebra on $\mathcal{H}^*(M) \cong H^*(M) \Rightarrow$ yielding an action of $SU(2)$ on $\mathcal{H}^*(M) \Rightarrow$ Hard Lefschetz Theorem:
  
  \[ L^l : H^{dM-l}(M) \xrightarrow{\cong} H^{dM+l}(M) \]
  
  \[ x \mapsto x \cup \alpha^l \]

  where $\alpha \in H^2(M)$ is the Kähler class.

- when $M$ is compact hyperkähler Hodge theory gives an action of an $N = 4$ supersymmetry algebra on $\mathcal{H}^*(M)$, which yields an action of $SO(5)$ on $\mathcal{H}^*(M)$

- What happens when $M$ is non-compact but complete?
Original Problem

Problem
What is the space $\mathcal{H}^*$ of $L^2$ harmonic forms on $\mathcal{M}_{\text{Dol}}$ the moduli space of Higgs bundles on a Riemann surface?

Motivation from physics

- (Sen 1994) conjecture predicts $\mathcal{H}^*$ for magnetic monopole moduli spaces
- (Vafa-Witten 1994) conjecture predicts $\mathcal{H}^*$ on the moduli of Yang-Mills instantons on ALE spaces

Hodge theory on complete Riemannian manifolds
⇒ have maps $H^*_c \rightarrow \mathcal{H}^* \rightarrow H^*$
⇒ $\text{Im}(H^*_c \rightarrow H^*)$ is a topological lower bound for $\mathcal{H}^*$

Theorem (Hausel 1998)
The intersection form $H^*_c(\mathcal{M}_{\text{Dol}}) \rightarrow H^*_{\text{mid}}(\mathcal{M}_{\text{Dol}})$ is trivial for rank two Higgs bundles. (Thus $\mathcal{H}^*(\mathcal{M}_{\text{Dol}})$ maybe trivial.)
Moduli of Higgs bundles

- $\Sigma$: genus $g$ Riemann surface
- $E \to \Sigma$: rank $n$ holomorphic bundle, $\phi : E \to EK$: Higgs field
  $(E, \phi)$: Higgs bundle
- $\mathcal{M}_{Dol}$: moduli space of rank $n$ stable Higgs bundles of degree 1
  smooth, quasi-projective variety, with a natural hyperkähler metric
- $(E, \phi)$: Higgs bundle - stable $\mapsto$
  $(d_E, \phi)$: unitary connection with Higgs field - solves Hitchin eqns $\mapsto$
  $d_E + \phi + \phi^*$: complex connection - flat $\mapsto$
  representation of $\pi_1(\Sigma) \to \text{GL}_n(\mathbb{C})$
Character variety

Non-Abelian Hodge theorem (Hitchin, Donaldson, Simpson, Corlette 1987-):

$$\mathcal{M}_{\text{Dol}} \cong_{\text{diff}} \mathcal{M}_B$$

where

$$\mathcal{M}_B := \text{Hom}(\pi(\Sigma \setminus \{p\}) \to \text{GL}(n, \mathbb{C})|_{\gamma_p \mapsto \exp(2\pi i/n)\text{Id}}) \backslash \text{GL}(n, \mathbb{C}) =$$

$$= \{A_1, B_1, \ldots, A_g, B_g \in \text{GL}(n, \mathbb{C})| A_1^{-1} B_1^{-1} A_1 B_1 \ldots A_g^{-1} B_g^{-1} A_g B_g = \exp(2\pi i/n)\text{Id}} \backslash \text{GL}(n, \mathbb{C})$$

is a smooth affine variety.
Weight filtration

- $X$ variety defined over $\mathbb{Z}$
- $Frob_q : H^k(X(\overline{F}_q); \mathbb{Q}_\ell) \to H^k(X(\overline{F}_q); \mathbb{Q}_\ell)$ Frobenius automorphism
- (Deligne 1974) proved Weil’s Riemann hypothesis: eigenvalues of $Frob_q$ have absolute value $q^{i/2}$ for $i \in \mathbb{N}$
- Jordan decomposition of $Frob_q$ on $H^k \Rightarrow$ weight filtration $W_l \subset H^k$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2} \leq l$
- comparison theorem: $H^*(X(\mathbb{C}); \mathbb{C}) \cong H^*(X(\overline{F}_q), \mathbb{Q}_\ell) \otimes \mathbb{C}$
- (Deligne 1972) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$ for any complex algebraic variety $X$, which is
  - functorial
  - compatible with cup-product
Example

- Take $X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \cong \{(x, y) \in \mathbb{C}^2 | xy = 1\}$
  
  $H^0(X; \mathbb{C}) \cong H^2_c(X; \mathbb{C}) \cong \mathbb{C}$, \( H^1(X, \mathbb{C}) \cong H^1_c(X, \mathbb{C}) \cong \mathbb{C} \)

- $X(\overline{\mathbb{F}}_q) = \overline{\mathbb{F}}_q^\times$

- $Frob_q : \overline{\mathbb{F}}_q^\times \to \overline{\mathbb{F}}_q^\times$
  
  $x \mapsto x^q$

- $X(\overline{\mathbb{F}}_q)^{Frob_q} = X(\mathbb{F}_q) = \mathbb{F}_q \setminus \{0\}$, thus $|X(\overline{\mathbb{F}}_q)^{Frob_q}| = q - 1$

- Grothendieck-Lefschetz $\Rightarrow |X(\overline{\mathbb{F}}_q)^{Frob_q}| = \sum_{i=0}^2 (-1)^i \text{Trace}(Frob_q : H^i_c(X, \mathbb{Q}_\ell) \to H^i_c(X, \mathbb{Q}_\ell))$

- thus $1 = Frob_q : H^1_c(X; \mathbb{Q}_\ell) \to H^1_c(X, \mathbb{Q}_\ell))$ and $q \cdot - = Frob_q : H^2_c(X; \mathbb{Q}_\ell) \to H^2_c(X, \mathbb{Q}_\ell))$

- $\Rightarrow$

  $0 = W_1(H^1(X(\mathbb{C}), \mathbb{Q})) \subset W_2(H^1(X(\mathbb{C}); \mathbb{Q})) = H^1(X(\mathbb{C}), \mathbb{Q})$
Arithmetic of character variety

For a smooth complex variety $X$ define $E$-polynomial
\[
E(X; q) = \sum W_i/W_{i-1}(H^k(X))(-1)^k q^{d-i}/2
\]

(Katz 2008) proves that if $E(q) := |X(\mathbb{F}_q)|$ is polynomial in $q$
then $E(X(\mathbb{C}); q) = E(q)$

(Hausel-Villegas 2008) calulates
\[
E(\mathcal{M}_B; q) = |\mathcal{M}_B(\mathbb{F}_q)| = \sum_{\chi \in Irr(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n)
\]

we find $E(\mathcal{M}_B; 1/q) = q^d E(\mathcal{M}_B; q)$ palindromic
by Alvis-Curtis duality
\[
q^{n(n-1)/2} \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in Irr(\text{GL}_n(\mathbb{F}_q))
\]

⇒ Curious Hard Lefschetz Conjecture:
\[
L^l : \text{Gr}_d^{W} (H^{i-l}(\mathcal{M}_B)) \xrightarrow{\chi} \text{Gr}_d^{W} (H^{i+l}(\mathcal{M}_B), \chi \cup \alpha^l)
\]
where $\alpha \in W_4 H^2(\mathcal{M}_B)$
Perverse filtration

- $f : X \to Y$ a \textit{proper} map between complex algebraic varieties of relative dimension $d$

- (de Cataldo-Migliorini 2005) introduce \textit{perverse filtration} $\subset P_i \subset P_{i+1} \subset \ldots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $f_!(\mathbb{Q}_X)$ into perverse sheaves

- the Relative Hard Lefschetz Theorem holds:

$$L^l : \ Gr^P_{d-l}(H^*(X)) \to Gr^P_{d+l}H^{*+2l}(X)$$

\[ x \mapsto x \cup \alpha^l \]

where $\alpha \in H^2(X)$ is a relative ample class
Main conjecture

- recall Hitchin map
  \[ \chi : \mathcal{M}_{Dol} \to \mathbb{A}^d \]
  \[ (E, \phi) \mapsto \text{charpol}(\phi) \]

- (Hitchin 1987) → completely integrable Hamiltonian system and proper

Conjecture

\[ P_k(\mathcal{M}_{Dol}) \cong W_{2k}(\mathcal{M}_B) \text{ under the isomorphism} \]
\[ H^*(\mathcal{M}_{Dol}) \cong H^*(\mathcal{M}_B) \text{ from non-Abelian Hodge theory} \]

- recipe (de Cataldo-Migliorini, ≥ 2008) for perverse filtration when \( X \) smooth and \( Y \) affine:
  take \( Y_0 \subset \cdots \subset Y_i \subset \cdots Y_d = Y \)
  s.t. \( Y_i \) generic with \( \text{dim}(Y_i) = i \) then
  \[ P_{k-i-1}H^k(X) = \ker(H^k(X) \to H^k(f^{-1}(Y_i))) \]

- thus Conjecture ⇒ ”topology of Hitchin map reflects the arithmetic of the character variety”
Universal Classes

▶ now on let $n = 2$, i.e. study rank 2 Higgs bundles
▶ $E \rightarrow \mathcal{M}_{\text{Dol}} \times \Sigma$ and $\Phi : E \rightarrow \mathbb{E}K$, universal Higgs bundle $(E, \Phi)|_{(E,\phi) \times \Sigma} = (E, \phi)$

▶

$$c_2(\text{End}(\mathbb{E})) = 2\alpha[\Sigma]^* + \sum_{i=1}^{2g} 4\psi_i e_i - \beta$$

for some $\alpha \in H^2(\mathcal{M})$, $\psi_i \in H^3(\mathcal{M})$ and $\beta \in H^4(\mathcal{M})$.
▶ (Hausel-Villegas 2008) $\Rightarrow \alpha, \psi_i, \beta \in W_4$,
▶ Conjecture $\Rightarrow \alpha, \psi_i, \beta \in P_2 \Rightarrow$
$\psi_i, \beta \in \ker(H^*(\mathcal{M}_{\text{Dol}}) \rightarrow H^*(\chi^{-1}(Y_0)))$
▶ Yes! was proved by (Thaddeus 1990)
▶ $\beta \in P_2 H^4(\mathcal{M}_{\text{Dol}})$ would mean
$\beta \in \ker(H^4(\mathcal{M}_{\text{Dol}}) \rightarrow H^4(\chi^{-1}(Y_1)))$ i.e. $\beta$ vanishes over a generic curve in $\mathbb{A}^d$. 
Hitchin’s heuristics for $\beta \in P_2H^4(M_{Dol})$

- need $\beta \in \ker(H^4(M_{Dol}) \to H^4(\chi^{-1}(Y_1)))$
- Around a singular fibre $\chi^{-1}(Y_1))$ topologically looks like $E \times T$ where
  - $T = T^{8g-8}$ is the Jacobian of the normalization of the singular spectral curve
  - $\tau : E \to \Delta \subset \mathbb{C}$ is an elliptic fibration the Tate curve: $\tau^{-1}\lambda \cong T^2$ for $\lambda \neq 0$ and $\tau^{-1}(0) = \text{nodal } \mathbb{P}^1$
  - "focus-focus singularity"
- calculate $c_2(M_{Dol}) = (2g - 2)\beta$
- note $c_2(M_{Dol})$ is a multiple of $\eta_S \in H^4(M_{Dol})$, where $S = \{x \in M_{Dol} | T\chi_x \text{ is not surjective}\}$ the singular locus of $\chi$
- note $S \cap E \times T = s \times T$
  where $s$ is the singular point on the nodal $\mathbb{P}^1 = \tau^{-1}(0) \subset E$
- thus $\eta_S$ can be moved to infinity on $\chi^{-1}(Y_1)$
- $\beta \in \ker(H^4(M_{Dol}) \to H^4(\chi^{-1}(Y_1)))$
Conclusion

- $\beta \in P_2 H^4(M_{Dol}) \Rightarrow \beta$ vanishes over a generic curve in $\mathbb{A}^d$
- $H^*(M_{Dol})$ generated by $\alpha, \psi_i, \beta$
- thus $\alpha, \psi_i, \beta \in P_2$ mirroring $\alpha, \psi_i, \beta \in W_4$
- the weight filtration is compatible with cup product \Rightarrow weight filtration on $H^*(M_{Dol})$ is determined
- our main Conjecture \Rightarrow the perverse filtration of the Hitchin map is compatible with cup-product! Why?
- (Ngo 2008) also studies the perverse filtration arithmetically on $M_{Dol}$ to get the fundamental Langlands lemma
- (Frenkel-Witten 2007) study the topology of the Hitchin map over generic curves in connection with the Geometric Langlands program and mirror symmetry
- Any connection to our considerations?